Definable linear orders in o-minimal structures

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Question

- Let M be an o-minimal group. Let $(P, \prec)$ be a M-definable total linear order. What does P look like?
- The simplest definable linear orders are the lexicographic ones on M^n. We use $<_{\text{lex}}$ to denote the lexicographic order.
  Obviously, a definable linear order can be a definable subset of such a lexicographic order, or the image of such a subset under a definable injection.
- Theorem: That's it.

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- Theorem: That's it.

Answer

Theorem A

Let M be an o-minimal field, let $(P, \prec)$ be an M-definable linear order with $n = \dim(P)$. Then there is an injection, $g : P \to M^{n+1}$, definable over the same parameters as P, such that g is an embedding of $(P, \prec)$ in $(M^{n+1}, <_{lex})$, and the projection of $g(P)$ to the last coordinate is finite.

Remark

We actually only need M to eliminate imaginaries and possess a definable order-reversing bijection from M to M. Then g maps P to M^{2n+1}, with finite projections to the odd coordinates.

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Prior Work

- Steinhorn has unpublished work that implies Theorem A when $\dim(P) = 1$.
- Steinhorn and Onshuus recently showed that a definable linear order could be broken up into finitely many pieces, on each of which Theorem A held.
- However, their result did not say how the order compared elements in different pieces, so the study of definable linear orders could not be reduced to the study of definable subsets of lexicographic orders.
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One-dimensional interleaving

Example

Let $P = (0,1) \cup (1,2)$, with the order $\prec$ defined to agree with $<$ on $(0,1) \times (0,1)$ and $(1,2) \times (1,2)$, and defined as $a \prec b$ iff $a \leq b - 1$ on $(0,1) \times (1,2)$.

$$.25 \prec .5 \prec 1.5 \prec .75 \prec 1.8 \prec 1.9$$

The embedding:

Send $a \in (0,1)$ to $\langle a, 0 \rangle$. Send $b \in (1,2)$ to $\langle b - 1, 1 \rangle$.

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Induction: pdim

Definition

For $x \in P$, let $\mathrm{pdim}(x) := \min\{\dim((y,z)_\prec) \mid x \in (y,z)_\prec\}$.

$\mathrm{pdim}(x)$ measures what the dimension of P is in a $\prec$-neighborhood of x.
For example, suppose that $P = \mathbb{R}^2$ and $\prec$ is the lexicographic order.

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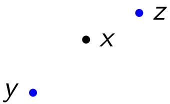

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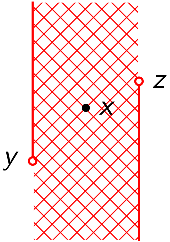

$\dim((y,z)_\prec) = 2$

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- z'
- x
- y'

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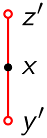

$$\dim((y',z')_\prec) = 1$$

An equivalence relation of closeness

Definition

For $x, y \in P$, let xEy if the $\prec$-interval bounded by x and y has dimension $< n$.

E is a $\prec$-convex equivalence relation on P.

Lemma

No E-class has dimension n.

By compactness – if not, take two "extreme" elements of an E-class. Then the $\prec$-interval they bound has dimension $< n$ but the intervening set has dimension n.

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Induction or else

Lemma pdim

If the set $\{x \in P \mid \mathrm{pdim}(x) = n\}$ *has dimension* $< n$*, then Theorem A follows.*

- The premise implies that P/E has dimension $< n$.
- By induction, P/E definably embeds in a lexicographic order. Also by induction, for each $x \in P/E$, the class $[x]_E$ definably embeds in a lexicographic order.
- With some careful stitching together while keeping track of dimensions, the theorem is proved.

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Or else

Lemma

$n \leq 1$.

- Let C be a cell with pdim $= n$ on C, witnessed "from below". We also require that the order be "continuous" on C.
- We follow a technique of Hasson and Onshuus, and pick a definable curve Γ in C.
- After restricting/redefining Γ, we may suppose that "$<$" and $\prec$ agree on Γ, and that Γ has endpoints.

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Fibers

- Let $T : P \to \Gamma$ be the function defined by $T(x) := \inf_{\Gamma,\prec}\{y \in \Gamma \mid y \succeq x\}$.
- By fiber arguments, the set $T^{-1}(y)$ has dimension $< n$ for all but finitely many $y \in \Gamma$, and we may restrict to some definable piece of Γ where $k = \dim(T^{-1}(y))$ is constant.
- Let $b \prec c$ be elements in this piece of Γ.
- Looking again at fibers,

$$(b,c)_\prec \subseteq \bigcup_{x \in [b,c]_\prec \cap \Gamma} T^{-1}(x),$$

so $n \leq 1 + k$.
- We want to show that $k = 0$.

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Bringing in pdim

- Suppose that $k > 0$. By hypothesis of continuity of $\prec$ on C, we can choose $d \in (b, c)_\prec \cap C \setminus \Gamma$. Let $a = T(d)$.
- Note that $(d, a)_\prec \subseteq T^{-1}(a)$ (or $(a, d)_\prec$ if $d \succ a$). Thus $\dim((d, a)_\prec) \leq \dim(T^{-1}(a)) \leq k < n$. But $\mathrm{pdim}(a) = n$, so $\dim((y, a)_\prec) < n$ for all y, contradiction.
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Partial orders

- We can ask if there exists a classification of definable partial orders, like the one that we have given for total orders.
- Clearly, this is much more difficult. For example, any family of definable sets is a partial order, with the order relation of inclusion.
- Nevertheless, we can ask:

Question

Let M be an o-minimal group, and let $(P, \prec)$ be a definable partial order. Does there exist a definable linear order $\prec'$ on P such that $\prec'$ extends $\prec$?

- It holds for dimension 1.

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When can pdim be bad?

- The proof showed that pdim is not n on an n-dimensional set for $n > 1$.
- Lemma pdim works in a much more general context than o-minimality.
- When is Lemma pdim true: for a definable linear order, if $\dim(\{x \mid \mathrm{pdim}(x) = n\}) < n$ then the order decomposes into a lexicographic product of lower-dimension orders?

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Structures with dimension

- Our proof uses the following properties of dimension:
  1. dim is a function from definable sets to $\mathbb{N}$.
  2. If E is a definable equivalence relation on a definable set X, and $\dim(X) = n$, then at most finitely many E-classes of X have dimension n.
  3. If E is a definable equivalence relation on a definable set X and $\dim(X) = \dim(X/E)$, then there is a definable $Y \subseteq X$ with $\dim(Y) = \dim(X)$ and for $y \in Y$ the set $[y]_E$ is finite.
  4. Dimension is uniformly definable – for any parametrically-definable set $X(y)$ and any natural number k, there is a formula $\theta^k(y)$ such that $\dim(X(y)) \geq k$ iff $\theta^k(y)$.
- Property (2) was first defined by Poizat and Pillay as a property of "chirurgical" structures.
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  4. Dimension is uniformly definable – for any parametrically-definable set $X(y)$ and any natural number k, there is a formula $\theta^k(y)$ such that $\dim(X(y)) \geq k$ iff $\theta^k(y)$.
- Property (2) was first defined by Poizat and Pillay as a property of "chirurgical" structures.
- All of these properties hold for pregeometric surgical structures, as defined by Gagelman.

Strange linear orders

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- In part, this is because the 1-dimensional set that we chose, Γ, may not be definably complete, and so the definition of T fails.
- Nevertheless, we may hope that it is true. Namely:

Question

Let P be a definable linear order in a pregeometric surgical structure. Does the set $\{x \in P \mid \mathrm{pdim}(x) = n\}$ always have dimension $< n$?

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