Definable linear orders in o-minimal structures

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7 Juin 2010

Logic Seminar, CMAF
http://www.janak.org/talks/lisbon-2010-07-01.pdf

Background

- The definition: an o-minimal structure is an ordered structure, M, such that every definable subset of M^1 is the union of finitely many points and intervals.
- In o-minimality, we have the cell decomposition theorem, which allows us to partition definable sets into finitely many cells such that each cell has the definable properties that we want.
- Following from this, we have uniform finiteness – for a definable family of finite sets, there exists a finite bound on the size of the sets in this family.

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Question

- Let M be an o-minimal group. Let $(P, \prec)$ be a total M-definable linear order. What does P look like?
- The simplest definable linear orders are the lexicographic ones on M^n. We use $<_{\text{lex}}$ to denote the lexicographic order.
  Obviously, a definable linear order can be a definable subset of such a lexicographic order, or the image of such a subset under a definable injection.
- Theorem: That's it.

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Answer

Theorem A

Let M be an o-minimal group, let $(P, \prec)$ be an M-definable linear order with $n = \dim(P)$. Then there is an injection, $g : P \to M^{2n+1}$, definable over the same parameters as P, such that g is an embedding of $(P, \prec)$ in $(M^{2n+1}, <_{lex})$, and the projections of $g(P)$ to the odd coordinates are finite.

Remark

In fact, we only need M to eliminate imaginaries and have a definable order-reversing bijection from M to M.

Remark

When M expands an ordered field, we can reduce the dimension of the target space to $n + 1$, with finite projection only to the last coordinate.

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Prior Work

- Steinhorn has unpublished work that implies Theorem A when $\dim(P) = 1$.
- Steinhorn and Onshuus recently showed that a definable linear order could be broken up into finitely many pieces, on each of which Theorem A held.
- However, their result did not say how the order compared elements in different pieces, so the study of definable linear orders could not be reduced to the study of definable subsets of lexicographic orders.
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One-dimensional interleaving

Example

Let $P = (0,1) \cup (1,2)$, with the order $\prec$ defined to agree with $<$ on $(0,1) \times (0,1)$ and $(1,2) \times (1,2)$, and defined as $a \prec b$ iff $a \leq b - 1$ on $(0,1) \times (1,2)$.

$$.25 \prec .5 \prec 1.5 \prec .75 \prec 1.8 \prec 1.9$$

The embedding:

Send $a \in (0,1)$ to $\langle a, 0 \rangle$. Send $b \in (1,2)$ to $\langle b-1, 1 \rangle$.

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General position

Example

Let $M = (\mathbb{R}, <, +, 0)$ and let $n > 0$. Let $P = \{\langle x_1, \ldots, x_{2n+1}\rangle \in M^{2n+1} \mid x_i \in \{0,1\}$ for i odd$\}$. Let $\prec$ be the lexicographic order on P.

P is already embedded in M^{2n+1}, but there is no definable embedding in $(M^m, <_{\text{lex}})$ for any $m < 2n+1$.

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Why $2n+1$?

This example helps to show why we need dimension $2n+1$ – there may be only finitely many possibilities for coordinate k, say $\{1,\ldots,r\}$, with the next coordinate lying in an unbounded interval.

With a field, we can map that unbounded interval into finitely many bounded disjoint intervals, one around each of $1,\ldots,r$.

But with a group, an unbounded interval cannot be mapped into a bounded interval, so we cannot perform this procedure.

On the other hand, if there are two consecutive coordinates with finitely many possibilities, the group can take them into one coordinate, accounting for the alternation.

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One dimension: "monotonicity" for orders

Lemma (Steinhorn, Onshuus and Steinhorn)

Let M be an o-minimal group and let $(P, \prec)$ be a definable linear order with $\dim(P) = 1$. Then P is definably isomorphic to a finite union of cells on each of which the induced $<$ and the induced $\prec$ agree everywhere

For this lemma, and throughout the proof, it helps to note that, if X is a definable family of sets, parametrized by a set A, we can partition A such that on each subset, the family varies continuously.

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One dimension: strategy

- After applying the monotonicity lemma, we have $P = I_1 \cup \ldots \cup I_k$, where each I_j is a 0- or 1-dimensional cell, and $\prec$ and $<$ agree on each I_j.
- By induction, we can suppose that $I_1 \cup \ldots \cup I_{k-1}$ can be mapped to P', a definable subset of M^3 ordered lexicographically, and our task is to insert I_k.
- We can partition I_k into "well-behaved" pieces, relative to P', and insert them one by one, keeping the remaining pieces "well-behaved" with respect to our new P'.

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Partitioning I_k

- Each point of I_k is in some Dedekind cut of P'.
- Either the cut is just after an element of P', or just before, or it is a "dense" cut.
- We would like to partition I_k into pieces such that for each piece, either every element is just after or just before an element of P', or every element is in the same cut of P'.
- We must show that each cut can contain either a single element of I_k or an infinite number of elements, and there are only finitely many cuts in the second case.

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n dimensions: Two dimension-counters

Definition

For $x \in P$, let $\operatorname{pdim}(x) := \min\{\dim((y,z)_\prec) \mid x \in (y,z)_\prec\}$.

$\operatorname{pdim}(x)$ measures what the dimension of P is in a $\prec$-neighborhood of x.

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For $x, y \in P$, let xEy if the $\prec$-interval bounded by x and y has dimension $< n$.

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E-classes and pdim cells

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No E-class has dimension n.

If there were, we would have a definable n-dimensional $\prec$-convex set such that any $\prec$-interval inside it had dimension $< n$. An argument shows that this cannot happen.

Let $\mathcal{C}$ be a cell decomposition of P such that on each cell $C \in \mathcal{C}$, we have constant pdim.

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Induction or else

Lemma

If every open cell in $\mathcal{C}$ has pdim $< n$*, then Theorem A follows.*

- The premise implies that P/E has dimension $< n$.
- By induction, P/E definably embeds in a lexicographic order. Also by induction, for each $x \in P/E$, the class $[x]_E$ definably embeds in a lexicographic order.
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Or else

We know that there is an open cell, C, with pdim $= n$ on C.

Lemma

$n \leq 1$.

We follow a technique of Hasson and Onshuus, and pick a definable curve Γ in C. After restricting/redefining Γ, we may suppose that "$<$" and $\prec$ agree on Γ.

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Fibres

- Let $T : P \to \Gamma$ be a partial function, defined by $T(x) := \inf_{\prec}\{y \in \Gamma \mid y \succeq x\}$.
- $T(x)$ is the smallest element in Γ at least as big as x.
- By fiber arguments, the set $T^{-1}(y)$ has dimension $< n$ for all but finitely many points of Γ, and we may restrict to some definable piece of Γ where $k = \dim(T^{-1}(y))$ is constant.
- Let $b \prec c$ be elements in this piece of Γ.
- Looking again at fibers,

$$n = \dim((b,c)_{\prec}) \leq \bigcup_{x \in (b,c]_{\prec} \cap \Gamma} T^{-1}(x) = 1 + k.$$

- We want to show that $k = 0$.

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Bringing in pdim

- Let $a \in (b, c)_\prec \cap \Gamma$, and suppose that $k > 0$. Then we can choose $d \in T^{-1}(a) \setminus \{a\}$.
- Note that $(d, a)_\prec \subseteq T^{-1}(a)$. Thus $\dim((d, a)_\prec) \leq \dim(T^{-1}(a)) \leq k < n$. But $a \in C$, so $\dim((y, a)_\prec) < n$ for all y, contradiction.
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Results in Economics

- The connections between this topic and economics are very strong.
- An article by Beardon et al (2002), "Lexicographic decomposition of chains and the concept of a planar chain" in the **Journal of Mathematical Economics** contains an idea very close to a central idea of this proof.
- When a linear order has a suborder that is cofinal, coinitial, and Dedekind complete, the order can be decomposed with an equivalence relation.
- After several repetitions, one shows the existence of an embedding of one piece of the order in $(\mathbb{R}^2, <_{\text{lex}})$.
- The procedure is extremely non-definable, and it is not clear how to decide if an order has such a suborder.
- They only consider embeddings in $(\mathbb{R}^2, <_{\text{lex}})$.

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Results in Economics

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Searching for utility functions

- For many years, economists thought that all orders embedded in $\mathbb{R}$. In 1954, Dubreu pointed out to them the obvious fact that the lexicographic order on $\mathbb{R}^2$ does not embed in $\mathbb{R}$. An order that does not embed in $\mathbb{R}$ is called "non-representable".
- Beardon et al give an analysis of non-representable orders: if a non-representable order does not contain ω_1 or ω_1^*, and has a representable uncountable suborder, then there is a non-representable uncountable suborder that embeds in $(\mathbb{R}^2, <_{\text{lex}})$.

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- We can ask if there exists a classification of definable partial orders, like the one that we have given for total orders.
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