Definable linear orders in o-minimal fields

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Question

- Let M be an o-minimal field. Let $(P, \prec)$ be an M-definable total linear order. What does P look like?
- The simplest definable linear orders are the lexicographic ones on M^n. We use $<_{\text{lex}}$ to denote the lexicographic order.
- Obviously, a definable linear order can be a definable subset of such a lexicographic order, or the image of such a subset under a definable injection.
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Answer

Theorem A

Let M be an o-minimal field, and let $(P, \prec)$ be an M-definable linear order with $n = \dim(P)$. Then there is an injection, $g : P \to M^{n+1}$, definable over the same parameters as P, such that g embeds $(P, \prec)$ in $(M^{n+1}, <_{lex})$, and the projection of $g(P)$ on the $(n+1)$-st coordinate is finite.

Prior Work

- Steinhorn has unpublished work that implies Theorem A when $\dim(P) = 1$.
- Steinhorn and Onshuus recently showed that a definable linear order could be broken up into finitely many pieces, on each of which Theorem A held.
- They also noted that such a result has applications in economics.
- However, their result did not say how the order compared elements in different pieces, so the study of definable linear orders could not be reduced to the study of definable subsets of lexicographic orders.

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One-dimensional interleaving

Example

Let $P = (0,1) \cup (1,2)$, with the order $\prec$ defined to agree with $<$ on $(0,1) \times (0,1)$ and $(1,2) \times (1,2)$, and defined as $a \prec b$ iff $a \leq b - 1$ on $(0,1) \times (1,2)$.

$$.25 \prec .5 \prec 1.5 \prec .75 \prec 1.8 \prec 1.9$$

The embedding:

Send $a \in (0,1)$ to $\langle a, 0\rangle$. Send $b \in (1,2)$ to $\langle b-1, 1\rangle$.

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Why $n + 1$

The example shows why we need the $(n + 1)$-st dimension – there can be finitely many pieces that are “interleaving”. O-minimality guarantees that there are not infinitely many.

One dimension: "monotonicity" for order

Lemma (Steinhorn, Onshuus and Steinhorn)

Let $(P, \prec)$ be an M-definable linear order, with $P \subseteq M$. Then P can be partitioned into finitely many points and intervals on each of which $\prec$ and $<$ either agree everywhere or disagree everywhere.

One dimension: strategy

- After applying the monotonicity lemma, and with some definable reversing maps (coming from the field), we will have $P = I_1 \cup \ldots \cup I_k$, with each I_j a point or an interval, and $\prec$ and $<$ agreeing on each I_j.
- By induction, we suppose that $I_1 \cup \ldots \cup I_{k-1}$ can be mapped to P', a definable subset of M^2 ordered lexicographically, and our task is to insert I_k.
- We can break I_k up into "well-behaved" pieces, relative to P', and insert them one by one, keeping the remaining pieces "well-behaved" with respect to our new P'.

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Dimension n: Two dimension-counters

Definition

For $x \in P$, let $\mathrm{pdim}(x) = \min\{\dim((y,z)_\prec) \mid x \in (y,z)_\prec\}$.

$\mathrm{pdim}(x)$ measures what the dimension of P is in a $\prec$-neighborhood of x.

Definition

For $x, y \in P$, let xEy if the $\prec$-interval bounded by x and y has dimension $< n$.

E is a $\prec$-convex equivalence relation on P.

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E-classes and pdim cells

Lemma

No E-class has dimension n.

If there were, we would have a definable n-dimensional $\prec$-convex set such that any $\prec$-interval inside it had dimension $< n$. An argument shows that this cannot happen.

Let $\mathcal{C}$ be a cell decomposition of P such that on each cell $C \in \mathcal{C}$, we have constant pdim.

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Induction or else

Lemma

If every open cell in $\mathcal{C}$ has $\mathrm{pdim} < n$*, then Theorem A follows.*

- The premise implies that P/E has dimension $< n$.
- By induction, P/E definably embeds in a lexicographic order. Also by induction, for each $x \in P/E$, the class $[x]_E$ definably embeds in a lexicographic order.
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Or else

We know that there is an open cell, C, with pdim $= n$ on C.

Lemma

$n \leq 1$.

We follow a technique of Hasson and Onshuus, and pick a definable curve Γ in C. After restricting/redefining Γ, we may suppose that "$<$" and $\prec$ agree on Γ.

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Fibers

- Let $T : P \to \Gamma$ be defined by $T(x) = \inf_{\prec}\{y \in \Gamma \mid y \succeq x\}$ and $B : T \to \Gamma$ by $B(x) = \sup_{\prec}\{y \in \Gamma \mid y \preceq x\}$.
- $T(x)$ is the least element in Γ which is at least as big as x and $B(x)$ is the greatest element in Γ which is at most as big as x.
- By fiber arguments, $T^{-1}(y)$ and $B^{-1}(y)$ have dimension $< n$ for all but finitely many points of Γ, and we may restrict to some piece of Γ where $k_1 = \dim(T^{-1}(y))$ and $k_2 = \dim(B^{-1}(y))$ are constant.
- Let $b \prec c$ be elements in this piece of Γ.
- Looking again at fibers, $n = \dim((b, c)_{\prec}) \leq 1 + k_1, 1 + k_2$.
- We want to show that at least one of k_1, k_2 is 0.

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Bringing in pdim

- Consider any $a \in (b, c)_{\prec} \cap \Gamma$. If $k_1, k_2 \neq 0$, we can take $d \in T^{-1}(a)$ and $e \in B^{-1}(a)$ with $d, e \neq a$.
- Note that $(d, a)_{\prec} \subseteq T^{-1}(a)$, and $(a, e)_{\prec} \subseteq B^{-1}(a)$. Thus $\dim((d, e)_{\prec}) \leq \dim(T^{-1}(a) \cup B^{-1}(a)) \leq \max(k_1, k_2) < n$. So $\text{pdim}(a) < n$, contradiction.
- Thus, one of k_1 or k_2 must be 0, so $\dim(P) = n \leq 1 + 0$.

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