Extending functions continuously in o-minimal structures

Janak Ramakrishnan

November 3, 2008

Extending Functions to Closures

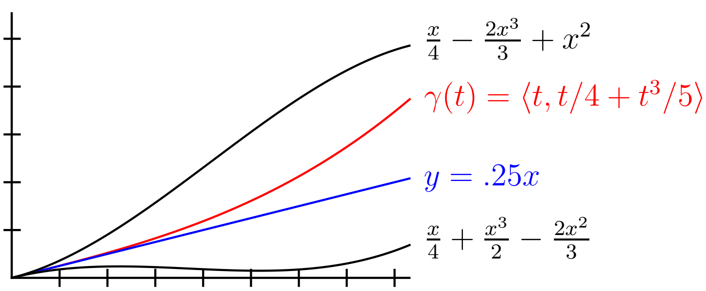
Let γ be a curve in M^n with one endpoint the origin, and let f be an M-definable bounded n-ary function. Can we find an initial segment of γ and a definable set containing that initial segment on which f is continuous, or extends continuously?[1]

Note that we can certainly find a definable set containing $\gamma \setminus \{0\}$ on which f is continuous. The difficulty is in extending f continuously to 0, which is equivalent to extending f continuously to the closure of the definable set.

Definable γ works

Example. Let $f(x, y) = \min(1, y/x)$, and let γ be any definable curve in the first quadrant with left endpoint 0.

We can take a pair of cubics whose derivatives at 0 are the same as the curve's at 0, giving us a cell on which f extends continuously to the closure.



Non-definable curves

What about the question for non-definable curves? Given a "well-behaved" non-definable curve, can we find a set on which the function is continuous, which contains the curve, and on whose closure the function extends continuously.

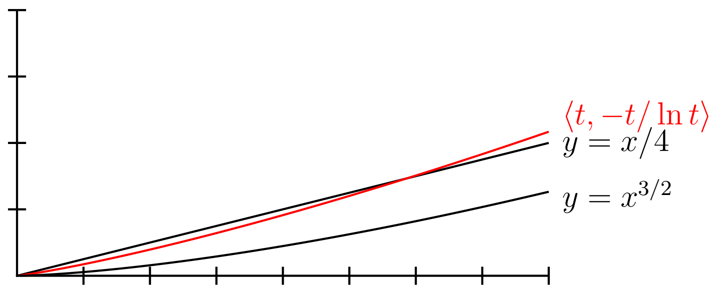
Counterexample with a non-definable curve

Let $M = (\mathbb{R}, +, \cdot, <, 0, 1)$. Let $f(x, y)$ be $\min(1, y/x)$, and let $\gamma(t) = \langle t, -t/\ln t \rangle$, so γ is undefinable in M. Since γ is definable in the o-minimal expansion of M, $(\mathbb{R}, +, \cdot, <, \exp)$, γ is certainly "well-behaved".

[1] Patrick Speissegger raised this question with me, and parts of the following work were done first by him in a different form, particularly one of the cases in the main proposition.

$f(\gamma(t)) = -1/\ln t$, so $\lim_{t\to 0^+} f(\gamma(t)) = 0$.

- $-1/\ln t$ goes to 0, but it is also greater than t^d, for any $d > 0$, for sufficiently small t. Thus, $-t/\ln t$ is greater than t^{1+d}, but less than at, for every $a \in \mathbb{R}_+$.
- It is not hard to see that any definable set in $(\mathbb{R}, +, \cdot, <, 0, 1)$ that contains γ must contain the curve $\langle t, at\rangle$, for some real positive a, as well as the curve $\langle t, bt^{1+q}\rangle$, for some real positive b and positive rational q.
- f cannot be continuously extended onto this set's closure, because along the latter curve, its limit at the origin is 0, while along the linear curve, it is a.



Why did γ fail?

The failure of γ can be seen as coming from the fact that we could not squeeze γ sufficiently. The gap between a linear function and a higher-power function is too great. To more closely analyze this, we can abstract out the "type" of γ, by considering the type of an infinitesimal point on the curve, which is given by the following.

- For every $r > 0 \in \mathbb{R}_+$, $x_1 < r$ is in $\text{tp}(\gamma)$.
- For every $r \in \mathbb{R}_+$, $x_2 < rx_1$ is in $\text{tp}(\gamma)$.
- For every $r \in \mathbb{R}$, $q \in \mathbb{Q}_+$, $x_2 > rx_1^{1+q}$ is in $\text{tp}(\gamma)$.

Since we have equivalence of definable set membership for curves and their types, we can rephrase our failure with γ as follows:

Example. Take our model to be $(\mathbb{R}, +, \cdot, <, 0, 1)$. Let $p(x, y)$ be the type which says that x is greater than 0 but less than every real, and that y is less than rx, for any $r \in \mathbb{R}_+$, but greater than rx^{1+q}, for any $r \in \mathbb{R}$, $q \in \mathbb{Q}_+$. It is easy to see that these conditions generate a complete consistent type. Let f be as before, $\min(1, y/x)$.

There is no definable set, C, with $C \in p$, f continuous on C, and f extending continuously to $\overline{C}$.

If we let $\langle c_1, c_2\rangle \models p$, the problem here is that the pre-images of elements of $\mathbb{R}$ under $f(c_1, -)$ are coinitial at c_2 in $\mathbb{R}(c_1)$.

Scale

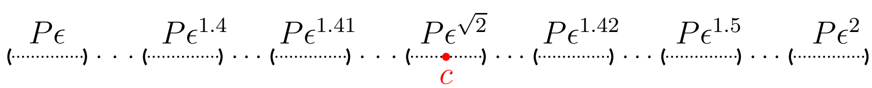
Definition (Marker)**.** For $A = \mathrm{acl}(A)$, $p \in S_1(A)$ is a *cut* iff it is non-algebraic and (1) there are formulas of the form $a < x$ and $x < a$ in p, and (2) for every formula of the form $a < x$ in p, there is $b > a$ such that $b < x$ is in p, and similarly for $x < a$. p is a *noncut* if it is non-algebraic and not a cut.

Definition (∼Marker-Steinhorn)**.** Let $A \subset B$, and $p \in S_1(B)$, with p a cut over B. Let c be any realization of p. If there is a B-definable unary function, f, such that $f(A)$ is both cofinal in B below c and coinitial in B above c, we say that p is *in scale on A*. Otherwise, if there is such an f with $f(A)$ cofinal or coinitial, but not both, we say that p is *near scale on A*. If no such f exists, we say that p is *out of scale on A*.

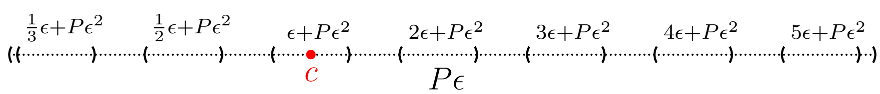
Scale examples

Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$. Let $N = M(\epsilon)$, where ϵ is infinitesimal. For notation, let $P = \mathbb{R}_+$.

1. If $c \models p = \mathrm{tp}(\epsilon^{\sqrt{2}}/N)$, then p is out of scale on M.

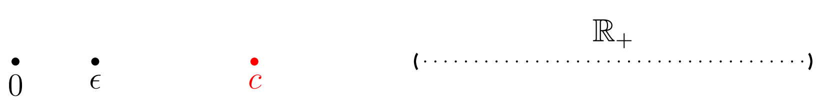


2. If $c \models p = \mathrm{tp}(\sum_{i=1}^{\infty} \epsilon^i/N)$, then p is out of scale on M.



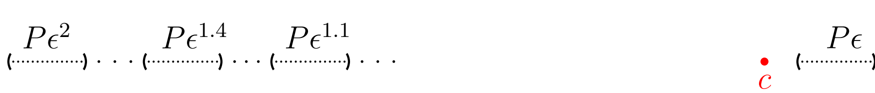
3. Let $M = (\mathbb{Q}^{\mathrm{rcl}}, +, \cdot, 0, 1, <)$, and let $N = M(\epsilon)$. If $c \models p = \mathrm{tp}(\pi\epsilon/N)$, then p is in scale on M since, if $f(x) = x\epsilon$, $f(M)$ is both cofinal and coinitial at c in N.

4. Let $M(\mathbb{R}, +, \cdot, 0, 1, <)$ and let $N = M(\epsilon)$. Let $c < P$, but larger than ϵ^d, for any rational $d > 0$.



   $\mathrm{tp}(c/N)$ is near scale on M since, if $f(x) = x$, $f(M)$ is coinitial at c in N. However, note that, if we take $N' = M(c)$, then ϵ is a noncut over N', so the scale issue does not arise.

5. Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$ and $N = M(\epsilon)$, and let c be smaller than $r\epsilon$ for $r \in \mathbb{R}_+$, but larger than ϵ^q for $q \in \mathbb{Q}_{>1}$.



   $\mathrm{tp}(c/N)$ is near scale on M since, if $f(x) = x\epsilon$, $f(M)$ is coinitial at c in N.

Leaping to conclusions

If we look at our examples, we see that, in addition to Example 5, Example 3, with $\langle \epsilon, \pi\epsilon \rangle$, is easily seen to have the same failure, with the same function of $\min(y/x, 1)$. So there are problems if a coordinate is near scale or in scale over the previous ones.

So perhaps each coordinate of the type being out of scale over the previous ones is the necessary criterion.

But Example 4 shows that we must be more careful – while $\langle \epsilon, c \rangle$ has the second coordinate near scale over the first, if we reverse the coordinates, $\langle c, \epsilon \rangle$ is just one infinitesimal followed by another, and it is not hard to show such a type cannot yield a counterexample.

Since order matters as to the scale of a coordinate of a type over the previous ones, our goal is to give a presentation of the type that will enable us to examine whether one coordinate is out of scale over the previous ones without having the rug pulled out from under us via a reordering.

Decreasing types

Definition. Let A be a set. Define $a \prec_A b$ iff there exists $a' \in \mathrm{dcl}(aA)$ such that $a' > 0$, and $(0, a') \cap \mathrm{dcl}(bA) = \emptyset$. Define $a \sim_A b$ if $a \not\prec_A b$ and $b \not\prec_A a$. Finally, let $a \precsim_A b$ if $a \sim_A b$ or $a \prec_A b$.

This definition captures the idea that a is infinitesimal relative to b over A, or at least that some element of $\mathrm{dcl}(Aa)$ is.

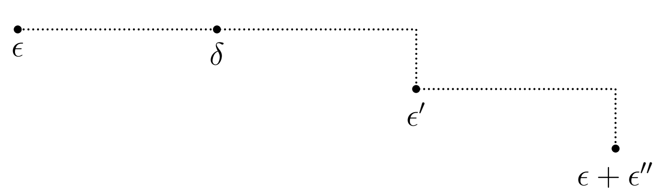
Lemma. *$\sim_A$ is an equivalence relation, and $\prec_A$ totally orders the $\sim_A$-classes.*

Notation. Assume that we have a fixed sequence $c = \langle c_i \rangle_{i \in I}$. Then the $\prec_i$-ordering is the $\prec_{c_{<i}}$-ordering. If we also have a fixed base set, A, then it will be the $\prec_{Ac_{<i}}$-ordering.

Definition. Let $p(x_1, \ldots, x_n) \in S_n(A)$. p is *decreasing* if, for some (any) realization, $c = \langle c_1, \ldots, c_n \rangle$ of p, $c_j \precsim_i c_i$, for $j > i$.

Lemma. *Any n-type can have its coordinates reordered so that it is decreasing.*

Example. Consider the tuple $\langle \epsilon, \epsilon + \epsilon'', \delta, \epsilon' \rangle$, where $1 \gg \epsilon \gg \epsilon' \gg \epsilon''$, and δ is a cut over ϵ. The tuple can be reordered as $\langle \epsilon, \delta, \epsilon', \epsilon + \epsilon'' \rangle$, which is decreasing.



Main result

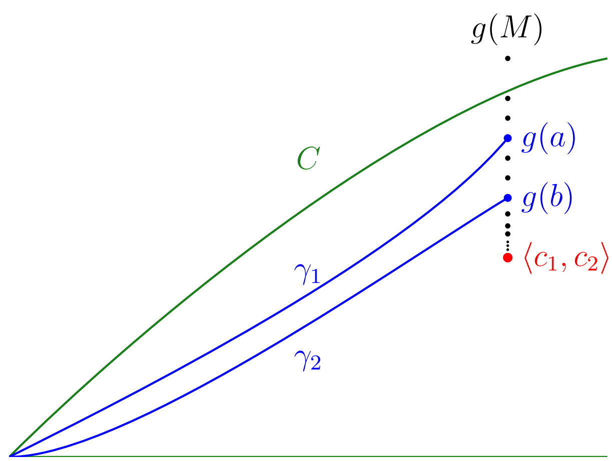
Theorem. *Let M be an o-minimal structure expanding a real closed field. Let $p \in S_n(M)$ be a decreasing type "near" the origin. Then the following two conditions are equivalent:*

1. *For $c = \langle c_1, \ldots, c_n \rangle$, some (any) realization of p, $\mathrm{tp}(c_i/c_{<i}M)$ is a noncut, or out of scale on M, for $i = 1, \ldots, n$.*
2. *For every M-definable function, f, bounded on some M-definable set in p, there is an M-definable set, C, in p, such that f is continuous on C and extends continuously to $\mathrm{cl}(C)$.*

Sketch of Backward Proof

The backward direction is fairly straightforward. Suppose that we have failure of the first condition. Then, at some coordinate, say the last one, we have some $Mc_{<n}$-definable function, g, such that $g(M)$ is near scale or in scale on M at c_n.

Consider $f = g^{-1}$ as a function of $c_{<n}$ and x. If C is any definable set containing c, we can choose $a \neq b \in M$ such that $g(a), g(b) \in C_{c_{<n}}$, and then, letting γ_1 and γ_2 be curves given by taking the pre-images of a and b under f, we get that it is impossible for f to extend continuously to the closure of C.

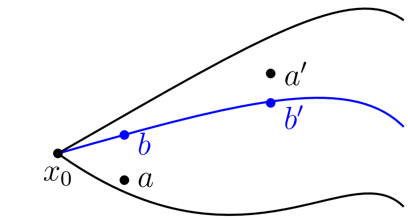


$f(x) = a$ for $x \in \gamma_1$, $f(x) = b$ for $x \in \gamma_2$.

Sketch of forward proof

For the forward direction, the proof works backwards along the coordinates of p. The auxiliary induction assumption that we use is that, when a and a' are tuples that agree through the ith coordinate, $|f(a) - f(a')|$ is bounded by a function that goes to 0 as the last coordinate that was a noncut goes to its limit.

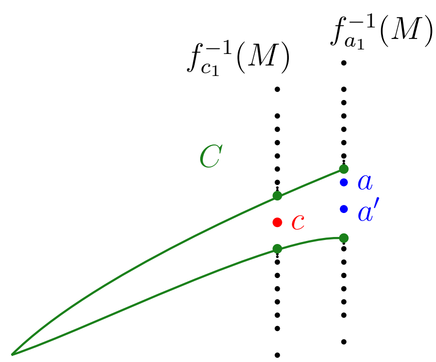
This ensures that, when the ith coordinate is a noncut, we can continuously extend f to the closure point. To maintain the above induction assumption, we can choose a definable curve in our set, and further restrict our set so that f applied to the curve stays very "close" to the limit value of f on the curve. Then, given two points that agree on their first $i-1$ coordinates, find a point on the curve that agrees with them on their first i coordinates, and use a triangle inequality:



$|f(b)-f(x_0)|$, $|f(b')-f(x_0)|$, $|f(a)-f(b)|$, $|f(a')-f(b')|$ all small.

Cut case

The noncut case is the one where any difficulties can lead to failure. The cut case is where difficulties start – where we may fail at preserving the induction assumption. We will have to ensure that two points, a and a', that agree up to their ith coordinates, will give similar values when f is applied to them. By doing the "opposite" of what was done in the proof of the backward direction, we can restrict to an interval that does not have any points from $f^{-1}(M)$. From that, one can prove that two points with ith coordinates in that interval are "close enough" when f is applied to them, using results about decreasing types.



$$\mathrm{tp}(f(a)/M) = \mathrm{tp}(f(a')/M).$$

(Surprisingly, also $\mathrm{tp}(f(a)/Ma_{<k(i)}) = \mathrm{tp}(f(a')/Ma_{<k(i)})$.)

Conclusion

With the theorem, our original case of a curve is resolved, by taking the curve's limit type.

While in this case, we were restricted from taking types that were interdefinable with our original, in circumstances where one can (for example, when examining definability), decreasing types allow for tighter results, since all near scale and in scale types can be removed – even our Example 5 disappears.