Hodges answers, Chapters 2.1-2.6, 3.1-8.4, 9.1-9.4

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2.1.1. By exhibiting suitable formulas, show that the set of even numbers is a Σ_0^0 set in $\mathbb{N}$. Show the same for the set of prime numbers.

$\exists y < x\,(2y = x)$. $\forall y < x\,(y = 1 \vee \forall z < x\,(yz \neq x))$.

2.1.2. Let A be the partial ordering (in a signature with $\leq$) whose elements are the positive integers, with $m \leq^A n$ iff m divides n. (a) Show that the set $\{1\}$ and the set of primes are both $\emptyset$-definable in A. (b) A number n is square-free if there is no prime p such that p^2 divides n. Show that the set of square-free numbers is $\emptyset$-definable in A.

(a) $\forall y\,(y \leq x)$. $\forall y\,(y \leq x \to y = 1 \vee y = x)$.

(b) Define the squares of primes by $\forall y\,(y \leq x \to y = 1 \vee y = x \vee \mathit{prime}\,(y)) \wedge \neg \mathit{prime}\,(x)$. Then the set of square-free numbers is $\forall y\,(y \leq x \to \neg \mathit{primesquared}\,(y))$.

2.1.3. Let A be the graph whose vertices are all the sets $\{m, n\}$ of exactly two natural numbers, with a joined to b iff $a \cap b \neq \emptyset$, $a \neq b$. Show that A is not minimal, but it has infinitely many minimal subsets.

Fixing any a, both the sets $R\,(x, a)$ and $\neg R\,(x, a)$ are infinite. However, if $a = \{m, n\}$, $b = \{m, k\}$, and $c = \{m, l\}$, then the set $R\,(x, a) \wedge R\,(x, b) \wedge R\,(x, c)$ is minimal, since given any formula $\theta\left(x, \bar{d}\right)$ whose intersection with this set is infinite and co-infinite, we can find an automorphism of A which fixes m and $\bar{d}$, but moves elements in $\theta\left(x, \bar{d}\right)$ out of it, and vice versa, which is impossible, since automorphisms preserve formulas.

2.1.4. Let A be a linear ordering with the order type of the rationals. Show that A is O-minimal.

This is trivial with quantifier elimination on dense linear orders without endpoints. Otherwise, given a formula $\varphi\,(x, \bar{a})$ defining some set in A, we can find an automorphism of A which fixes $\bar{a}$ pointwise and moves other points arbitrarily, subject to the constraint that if $a_i < x < a_j$, then $a_i < fx < a_j$. This shows that if Y has a point in the interval (a_i, a_j), then in fact it contains that interval. Thus, Y is a finite union of intervals along with zero or more of the a_i's.

2.1.5. Let A be an infinite-dimensional vector space over a finite field. Show that A is minimal, and that the only $\emptyset$-definable sets are $\emptyset$, $\{0\}$, and $\mathrm{dom}\,(A)$.

Let X be any set which is infinite and co-infinite, defined by $\varphi\,(x, \bar{a})$. X and X^C must both have elements in infinitely many dimensions. By considering the automorphism which takes every dimension which does not intersect $\bar{a}$ and multiplies it by λ, for λ in the finite field, we know that if there is an element of X in some dimension, then X contains that dimension entirely, and likewise for X^C. Thus, there are at least two dimensions disjoint from $\bar{a}$, such that the first is entirely in X and the second entirely in X^C. Taking an automorphism fixing $\bar{a}$ and mapping the first dimension to the second, we see that this is nonsense.

The three above sets are clearly $\emptyset$-definable, along with $\operatorname{dom}(A) \setminus \{0\}$. Since given any a and b non-zero there is always an automorphism taking a to b, these must be all of the $\emptyset$-definable sets.

2.1.6. (Time-sharing with formulas). Let $\bar{x}$ be a k-tuple of variables and let $\phi_i(\bar{x}, \bar{y})$ $(i < n)$ be formulas of a first-order language L. Show that there is a formula $\psi(\bar{x}, \bar{y}, \bar{w})$ of L such that for every L-structure A with at least two elements, and every tuple $\bar{a}$ in A, the set of all relations of the form $\psi\left(A^k, \bar{a}, \bar{b}\right)$ with $\bar{b}$ in A is exactly the set of all relations of the form $\phi_i\left(A^k, \bar{a}\right)$, with $i < n$.

Let $\bar{w} = (w_0, \ldots, w_n)$. Let $\psi\left(A^k, \bar{x}, \bar{y}\bar{w}\right)$ be

$$\left(\bigvee_{i<n} w_n = w_i \wedge \bigwedge_{j<n, j\neq i} w_n \neq w_j \wedge \phi_i(\bar{x}, \bar{y})\right) \vee \left(\bigwedge_{i<n} w_n \neq w_i \wedge \phi_0(\bar{x}, \bar{y})\right).$$

2.1.7. Show that if L is a first-order language then $|L|$ is equal to $\omega+$ (the number of symbols in the signature of L).

If $|L|$ is countable, the formulas $\exists_{=n} x\,(x = x)$ are all non-equivalent, so we are done. If L is uncountable, then the formulas $R(\bar{x})$, $f(\bar{x}) = y$, and $x = c$, for all of its symbols, are all non-equivalent and there are $|L| = \omega + |L|$ of them.

2.1.8. Prove Lemma 2.1.1 for formulas of $L_{\infty\omega}$ with finitely many free variables, by induction on the complexity of the formulas.

We show Lemma 2.1.1 where Y may be a relation on tuples of A. For an atomic formula, we have $R(\bar{t}(\bar{x}, \bar{a}))$ iff $\bar{x} \in Y$, for $\bar{t}$ some term of L (R can be "="). Then if an automorphism, f, fixes $\bar{a}$, since automorphisms preserve functions, constants, and relations, $\bar{t}$ is the result of functions and constants, and R is a relation, we have $R(\bar{t}(\bar{x}, \bar{a}))$ iff $\bar{x} \in fY$, so f fixes Y setwise. That takes care of the base case. Let $\varphi(\bar{x}, \bar{a})$ define Y. We must show that any automorphism of A fixing $\bar{a}$ pointwise fixes Y setwise. Suppose $\varphi(\bar{x}, \bar{a}) = \bigwedge_{i<\gamma} \psi_i(\bar{x}, \bar{a})$. Then by induction, the set defined by each ψ_i is fixed by the automorphism, and thus the intersection is fixed. Likewise for $\bigvee$ and the union. If $\varphi = \neg\psi$, since the set defined by ψ is fixed, the complement is also fixed. If $\varphi = \exists y \psi(\bar{x}, y, \bar{a})$, then by induction, the set defined by $\psi(x, y, \bar{a})$ is fixed by this automorphism. Thus, if $\left(\bar{b}, c\right)$ is in this set, its image, $\left(\bar{d}, e\right)$ is also in this set. Thus, if $\exists y \psi\left(\bar{b}, y, \bar{a}\right)$, then $\exists y \psi\left(f\bar{b}, y, \bar{a}\right)$. The same is true for $\forall$.

2.1.9. Let L be the signature of abelian groups and p a prime. Let A be the direct sum of infinitely many copies of $\mathbb{Z}\left(p^2\right)$, the cyclic group of order p^2. Show (a) the subgroup of elements of order $\leq p$ is $\emptyset$-definable and minimal, (b) the set of elements of order p^2 is $\emptyset$-definable but not minimal.

(a) The set is defined by $x + x + \ldots + x = 0$ (p copies of x). Note that this set corresponds to elements which have at least one coordinate equal to p, and all coordinates either p or 0. Suppose it is not minimal, say with an infinite co-infinite set, Y, defined by $\varphi(x, \bar{a})$. Each $a_i \in \bar{a}$ partitions ω into p^2 classes – the k-th coordinate of a_i determines the class of k. Thus, taking the intersection of these finitely many classes, we break ω up into some finite number of classes, say n, such that any mapping of the coordinates respecting those classes fixes $\bar{a}$. Now, let $y_1 \in Y$ and $y_2 \in Y^C$ be such that on each class, y_1 is not identically 0 or p on each coordinate, and y_2 is not identically 0 or p on each coordinate. This is possible because there are only finitely many possibilities, and both Y and Y^C are infinite. Then there is an automorphism of each class (the identity on the rest) which takes the element which is 1 on the coordinates where y_1 is p, and 0 otherwise, and maps it to the element which is 1 on the coordinates where y_2 is p, and 0 otherwise. This automorphism can be made to fix $\bar{a}$. Then the composition of all these automorphisms takes y_1 to y_2 and fixes $\bar{a}$, which is impossible.

(b) The set is defined by $\bigwedge_{n<p^2} nx \neq 0$. It is not minimal since if we fix some $a \in A$, we can consider all elements which are expressible as $a+$ a term of order $\leq p$. Since there are infinitely many such terms, this set is infinite, but it is also co-infinite, since $x - a$ will have order p^2 for infinitely many x.

2.1.10. Let G be a group, and let us call a subgroup H definable if H is first-order definable in G with parameters. Suppose that G satisfies the descending chain condition on definable groups of finite index in G. Show that there is a unique smallest definable subgroup of finite index in G. Show that this subgroup is in fact $\emptyset$-definable, and deduce that it is a characteristic subgroup of G.

Suppose there is no unique smallest definable subgroup of finite index in G. Considering the partial ordering of subgroups of G (ordered under reverse inclusion), by Zorn's lemma coupled with the descending chain condition, there is a smallest definable subgroup of finite index in G, H. We show that H is unique. Suppose not. Then there is some H' with $|H| = |H'|$ and H' a definable subgroup of finite index not equal to H. Then $H \cap H'$ contradicts the minimality of H and H', as the intersection of two subgroups of finite index has finite index.

Let $\varphi(x, \bar{a})$ define H. Let $\psi(x, \bar{y})$ be the formula $\varphi(x, \bar{y})$ along with the statement "$\varphi(x, \bar{y})$ defines a group of index $[G : H]$." This is first-order since H has finite index in G. Then $\psi(x, \bar{y})$ must define H or $\emptyset$ for all $\bar{y}$, since otherwise we would get a different group of minimal index, and could intersect it with H. Then we can define H by $\exists \bar{y} \varphi(x, \bar{y})$. Thus every automorphism fixes H, so it is a characteristic subgroup.

2.2.1. Let L be a first-order language and T a theory in L. Show: (a) if T and U are theories in L then $T \subseteq U$ implies $\mathrm{Mod}(U) \subseteq \mathrm{Mod}(T)$, (b) if $\mathbf{J}$ and $\mathbf{K}$ are classes of L-structures then $\mathbf{J} \subseteq \mathbf{K}$ implies $\mathrm{Th}(\mathbf{K}) \subseteq \mathrm{Th}(\mathbf{J})$, (c) $T \subseteq \mathrm{Th}(\mathrm{Mod}(T))$ and $\mathbf{K} \subseteq \mathrm{Mod}(\mathrm{Th}(\mathbf{K}))$, (d) $\mathrm{Th}(\mathrm{Mod}(T)) = T$ if and only if T is of the form $\mathrm{Th}(\mathbf{K})$, and likewise $\mathrm{Mod}(\mathrm{Th}(\mathbf{K})) = \mathbf{K}$ if and only if $\mathbf{K}$ is of the form $\mathrm{Mod}(T)$.

Let A be any model in $\mathrm{Mod}(U)$. Then $A \models U$, so $A \models T$, so A is in $\mathrm{Mod}(T)$. Let φ be any sentence in $\mathrm{Th}(K)$. Then φ is true in every model of $\mathbf{K}$, hence every model of $\mathbf{J}$, hence $\varphi \in \mathrm{Th}(\mathbf{J})$. Let φ be any sentence of T. Then for any $A \models T$, $A \models \varphi$, so $\varphi \in \mathrm{Th}(\mathrm{Mod}(T))$. Let A be any model of $\mathbf{K}$. Then $A \models \mathrm{Th}(\mathbf{K})$, so $A \in \mathrm{Mod}(\mathrm{Th}(\mathbf{K}))$. Suppose $T = \mathrm{Th}(\mathrm{Mod}(T))$. Then let $\mathbf{K} = \mathrm{Mod}(T)$. Suppose $T = \mathrm{Th}(\mathbf{K})$. Then $\mathrm{Mod}(T) \supseteq \mathbf{K}$, so $\mathrm{Th}(\mathbf{K}) \supseteq \mathrm{Th}(\mathrm{Mod}(T))$, so $T \supseteq \mathrm{Th}(\mathrm{Mod}(T))$, but the reverse inclusion holds from above. Suppose $\mathrm{Mod}(\mathrm{Th}(K)) = K$. Then let $T = \mathrm{Th}(\mathbf{K})$ and $K = \mathrm{Mod}(T)$. Conversely, if $K = \mathrm{Mod}(T)$, then $\mathrm{Th}(\mathbf{K}) \supseteq T$, so $\mathrm{Mod}(\mathrm{Th}(K)) \subseteq \mathrm{Mod}(T) = K$, and the reverse inclusion holds from above.

2.2.2. Let L be a first-order language and for each $i \in I$ let $\mathbf{K}_i$ be a class of L-structures. Show that $\mathrm{Th}\left(\bigcup_{i\in I} \mathbf{K}_i\right) = \bigcap_{i\in I} \mathrm{Th}(\mathbf{K}_i)$.

Let the theories be T and U. Suppose $\varphi \in T$. Then φ holds in every model of every $\mathbf{K}_i$. Thus, it is in U. The reverse is the same.

2.2.3. Let L be a first-order language and for each $i \in I$ let T_i be a theory in L. (a) Show that $\mathrm{Mod}\left(\bigcup_{i\in I} T_i\right) = \bigcap \mathrm{Mod}(T_i)$. In particular if T is any theory in L, $\mathrm{Mod}(T) = \bigcap_{\phi\in T} \mathrm{Mod}(\phi)$. (b) Show that the statement $\mathrm{Mod}\left(\bigcap_{i\in I} T_i\right) = \bigcup_{i\in I} \mathrm{Mod}(T_i)$ holds when I is finite and each T_i is of the form $\mathrm{Th}(\mathbf{K}_i)$ for some class $\mathbf{K}_i$.

Let $\mathbf{K}$ and $\mathbf{J}$ be the classes of models. If $A \in \mathbf{K}$, then $A \models T_i$ for each i, so $A \in \mathbf{J}$. The reverse is similar. For (b), let $\mathbf{K}$ and $\mathbf{J}$ be the classes of models. If $A \in \mathbf{J}$, then clearly $A \models \bigcap_{i\in I} T_i$, since $A \models T_i$ for some i, so $A \in K$. Now let $A \in \mathbf{K}$. Suppose A is not in $\mathrm{Mod}(T_i)$ for any i. Let φ_0 be a sentence such that $T_0 \vdash \varphi$ but $A \models \neg\varphi$. Then for some T_{i_1}, T_{i_1} does not prove φ. Let φ_1 be a sentence in T_{i_1} such that $A \models \neg\varphi_1$. Then find T_{i_2} such that T does not prove φ_2. Continuing this process, we will get $\varphi_0, \ldots, \varphi_n$ such that $A \models \neg\varphi_i$, for each $i \leq n$, but every theory T_j proves one of them. Then $\bigvee_{i\leq n} \varphi_i$ is in T_j for every $j \in I$, but false in A, which is impossible. Thus, $A \in \mathbf{J}$.

2.2.4. Let L be any signature containing a 1-ary relation symbol P and a k-ary relation symbol R. (a) Write down a sentence of $L_{\omega_1\omega}$ expressing that at most finitely many elements x have the property $P(x)$. (b) When $n < \omega$, write down a sentence of $L_{\omega\omega}$ expressing that at least n k-tuples $\bar{x}$ of elements have the property $R(\bar{x})$.

$\bigvee_{i<\omega} \forall x_0 \ldots x_{i-1} \left(\bigwedge_{j<i} P(x_j) \to \bigvee_{j<k<i} x_i = x_j\right)$. Let $\bar{x}(j)$ be the j-th component of $\bar{x}$ $(j < n)$. $\exists \bar{x}_0, \ldots, \bar{x}_{n-1} \left(\bigwedge_{i<k<n} \bigvee_{j<n} \bar{x}_i(j) \neq \bar{x}_k(j) \wedge \bigwedge_{i<n} R(\bar{x}_i)\right)$.

2.2.5. Let L be a first-order language an A a finite L-structure. Show that every model of $\mathrm{Th}(A)$ is isomorphic to A.

$\mathrm{Th}(A)$ certainly gives how many elements A has, so any model, B, has $|B| = |A|$. Let $n = |A|$. There are thus $n!$ maps to consider. We can fix $\bar{a} = (a_0, \ldots, a_{n-1})$, and just consider orderings of B, as $(b_0, \ldots, b_{n-1})$, with the mapping $b_i \to a_i$. Take a random such ordering. If it is an isomorphism, we are done. Otherwise, there is some atomic formula, which we can assume to be of the form $R_0(\bar{x})$, with $B \models R_0(\bar{b})$ but $A \models \neg R_0(\bar{a})$, or vice versa. Since $A \models \exists \bar{x} \left(\bigwedge_{i<n} x_i \neq x_j \wedge \neg R_0(\bar{x})\right)$, we can

restrict to orderings of B which witness this sentence. Take a new ordering witnessing this sentence and repeat. After at most $n!$ tries, we will have $A \models \exists \bar{x} \left(\bigwedge_{i<n} x_i \neq x_j \wedge \bigwedge_{i<n!} \neg R_i (\bar{x}) \right)$, but then B cannot satisfy this sentence, and so was not a model of $\mathrm{Th}(A)$ to begin with.

2.2.6. For each of the following classes, show that it can be defined by a single first-order sentence. (a) Nilpotent groups of class k ($k \geq 1$). (b) Commutative rings with identity. (c) Integral domains. (d) Commutative local rings. (e) Ordered fields. (f) Distributive lattices.

(a) The group axioms can obviously be written as a single sentence. The group C_i is defined to be $\forall y \exists c \, (c \in C_{i-1} \wedge xy = yxc)$, with C_0 $\forall z \, (xz = zx)$. Then each C_i is definable, so the group axioms together with $\forall x \, (x \in C_k)$ is a first-order sentence.

(b) Conjunct together the ring axioms together with $\forall xy \, (xy = yx)$.

(c) Conjunct together the commutative rings with identity axioms along with $\forall xy \, (xy = 0 \rightarrow y = 0 \vee x = 0)$.

(d) Conjunct together the commutative rings with identity axioms along with the axiom $\forall xyz \, (\neg \exists w \, (wx = 1 \vee wy = 1) \rightarrow (\neg \exists w \, (w \, (x + y) = 1) \wedge \neg \exists w \, (zx = 1)))$, saying that the set of non-units forms an ideal.

(e) Conjunct together the field axioms together with the linear order axioms together with $\forall xyz \, (y < z \rightarrow y + x < z + x)$ and $\forall xyz \, (x > 0 \wedge y < z \rightarrow x \cdot y < x \cdot z)$.

(f) Conjunct together the axioms for lattices, together with $\forall xyzw \, (x \cap (y \cup z) = (x \cap y) \cup (x \cap z))$.

2.2.7. For each of the following classes, show that it can be defined by a set of first-order sentences. (a) Divisible abelian groups. (b) Fields of characteristic 0. (c) Formally real fields. (d) Separably closed fields.

$\forall x \exists y \, (ny = x)$, for all $n > 0$. $n \neq 0$, for all $n > 0$. $\forall x_0 \ldots x_{n-1} \left(\sum_{i<n} x_i^2 \neq -1 \right)$, for all $n > 0$. We can say that in every algebraic extension, some element in the extension has a minimal polynomial of degree d such that for any algebraic extension of degree $\leq d!$, there are not d roots of this polynomial – see Exercise 4.3.8, which shows how to define an algebraic extension, how to define the minimal polynomial of an element in such an extension, and how to say that there are m roots of a polynomial in an extension, for any m.

2.2.8. Show that the class of simple groups is definable by a sentence of $L_{\omega_1 \omega}$.

The sentence is the conjunction of the theory of groups together with $\forall x \forall y \bigvee_{i<\omega} \exists z_0 \ldots z_{i-1} \left(y = \Pi_{i<n} z_i x z_i^{-1} \right)$.

2.2.9. Let L be a signature with a symbol $<$, and T the theory in L which expresses that $<$ is a linear ordering. (a) Define, by induction on the ordinal α, a formula $\theta_\alpha (x)$ of $L_{\infty\omega}$ which expresses (in any model of T) 'The order-type of the set of predecessors of x is α'. (b) Write down a set of axioms in $L_{\infty\omega}$ for the class of orderings of order-type α. Check that if α is infinite and of cardinality κ, your axioms can be written as a single sentence of $L_{\kappa^+\omega}$.

(a) Define $\theta_0(x)$ by $\neg\exists y(y<x)$. Given the formulas $\theta_\beta(x)$ for $\beta<\alpha$, define $\theta_\alpha(x)$ by $\forall y < x\bigvee_{i<\alpha}\theta_i(y)\wedge\forall yz<x\bigwedge_{i<\alpha}\theta_i(y)\wedge\theta_i(z)\rightarrow y=z$.

(b) The set of axioms is $\exists x\theta_\beta(x)$, for $\beta<\alpha$, and $\neg\exists x\theta_\alpha(x)$, along with the axioms for a linear order. The above sentences can be rewritten as $\forall x\bigvee_{i<\alpha}\theta_i(x)$, which is a sentence of $L_{\kappa^+\omega}$, since there are κ-many disjunctions in it.

2.3.1. Show that a theory T in a first-order language L is closed under taking consequences if and only if $T=\mathrm{Th}(\mathrm{Mod}(T))$.

If T is closed under taking consequences, let φ be any sentence in $\mathrm{Th}(\mathrm{Mod}(T))$. Since every $A\models T$ has $A\models\varphi$, φ is a consequence of T. Thus, $\varphi\in T$. The inclusion the other way is trivial. Conversely, if $T=\mathrm{Th}(\mathrm{Mod}(T))$, since $\mathrm{Th}()$ is closed under taking consequences, so is T.

2.3.2. Let T be the theory of vector spaces over a field K. Show that T is λ-categorical whenever λ is an infinite cardinal $>|K|$.

We construct an isomorphism from A to B, where A and B are K-vector spaces with cardinality λ. We use a back-and-forth game. Suppose we have $\left(\bar{a},\bar{b}\right)$ chosen so far, with $(A,\bar{a})\equiv_0\left(B,\bar{b}\right)$. The case when $\bar{a}$ has length 0 is trivial. Let $\forall$ choose c in A. If $c=\sum_{i<n}k_ia_i$ with $k_i\in K$, then set $d=\sum_{i<n}k_ib_i$. By 0-equivalence, $\bar{a}c$ and $\bar{b}d$ satisfy all the same quantifier-free formulas. If c is not such a sum, then choose d so that it is not any sum of the b_i's. This is possible since there are $|K|^n<\max(K,\omega)$ many possible sums, but $\lambda\geq(|K|,\omega)$ possible elements to choose from. Then it is easy to verify that $\bar{a}c$ and $\bar{b}d$ satisfy the same quantifier-free formulas (pretty much the same ones that $\bar{a}$ and $\bar{b}$ satisfied). This procedure can continue for λ many steps, at which point we have constructed an isomorphism between A and B.

2.3.3. Let $\mathbb{N}$ be the natural number structure $(\omega,0,1,+,\cdot,<)$; let its signature be L. Write a sentence of $L_{\omega_1\omega}$ whose models are precisely the structures isomorphic to $\mathbb{N}$.

$\forall xy\bigvee_{i,j<w}x=i\wedge y=j\wedge(x<y\leftrightarrow i<j)\wedge x+y=i+j\wedge xy=ij$.

2.3.4. Prove the lemma on constants (Lemma 2.3.2).

We must show that if $T\vdash\phi(\bar{c})$ then $T\vdash\forall\bar{x}\phi(\bar{x})$, where $\bar{c}$ is a tuple of constants not appearing in T. Let $A\models T$ be any model of T. Make A into a model of $L\cup\{\bar{c}\}$ by assigning $\bar{c}$ to elements of A arbitrarily. Then, since $T\vdash\varphi(\bar{c})$, $A\models\varphi(\bar{c})$. Since this is true for any assignment, $A\models\forall\bar{x}\varphi(\bar{x})$. Then since A was arbitrary, $\forall\bar{x}\varphi(\bar{x})$ is a consequence of T.

2.3.5. Show that if L is a first-order language with finitely many relations, functions and constant symbols, then there is an algorithm to determine, for any finite set T of quantifier-free sentences of L, whether or not T has a model.

T can be written as a single sentence, and hence as a disjunction of conjunctions of closed literals. We then need to check each conjunction, so we may reduce to the case that T is a single conjunction of closed literals. By Exercises 1.5.2 and 1.5.3, T has a model iff whenever $\neg\phi$ is a negated atomic sentence of T, then ϕ is not in the $=$-closure of the set of atomic sentences in T. It is then a simple matter to start constructing the $=$-closure, and stopping when we have enumerated all sentences of complexity at most ϕ, of which there are finitely many. If ϕ appears in this list, then T has no model, and otherwise, it does.

2.3.6. For each $n < \omega$ let L_n be a signature and Φ_n a Hintikka set for L_n. Suppose that for all $m < n < \omega$, $L_m \subseteq L_n$ and $\Phi_m \subseteq \Phi_n$. Show that $\bigcup_{n<\omega} \Phi_n$ is a Hintikka set for the signature $\bigcup_{n<\omega} L_n$.

All conditions (3.1-8) of Hintikka sets are clearly preserved under unions.

2.3.7. Let L be a first-order language. (a) Show that if there is an empty L-structure A and ϕ is a prenex sentence which is true in A, then ϕ begins with a universal quantifier. (b) Show (without assuming that every structure is non-empty) that every formula $\phi(\bar{x})$ of L is logically equivalent to a prenex formula $\psi(\bar{x})$ of L. (c) Sometimes it is convenient to allow l to contain 0-ary relation symboles (i.e. sentence letters) p; we interpret them so that for each L-structure A, p^A is either truth or falsehood, and in the definition of $\models$ we put $A \models p \Leftrightarrow p^A =$ truth. Show that in such a language L there can be a sentence which is not logically equivalent to a prenex sentence.

(a) By the definition of $\models$, if ϕ began with an existential quantifier, say was $\exists x\theta(x)$, then $A \models \phi$ would mean that for some $a \in A$, $A \models \theta(a)$, but there is no such a, so this is impossible.

(b) We wish to give a formula $\phi^*(\bar{x})$ which is prenex and such that in every model A with signature L and elements $\bar{a}$ from that model, $A \models \phi(\bar{a}) \Leftrightarrow A \models \phi^*(\bar{a})$. Since if $\bar{x}$ is not empty, A cannot be empty, the usual procedure works. If $\bar{x}$ is empty, then let ϕ^* be the usual prenex form. Let ϕ' be as follows. Let $\theta(\bar{x})$ be the quantifier-free formula in ϕ^*. Let $\theta'(\bar{x}, y)$ be $\theta(\bar{x}) \vee (y \neq y)$. Now replace θ in ϕ^* with θ'. If $A \models \phi$, then prepend $\forall y$ to ϕ^* to get ϕ'. If $A \models \neg\phi$, then prepend $\exists y$. Then ϕ' is in prenex normal form and agrees with ϕ on all models.

(c) Let φ be the formula $p \leftrightarrow \forall x (R(x))$, with L having a unary relation R. Let A be an empty model with $A \models p$ and B be an empty model with $B \models \neg p$. Then φ is true in A and false in B. Since a prenex sentence in an empty model is (by definition) true iff it begins with a universal quantifier, if φ has a prenex form, it is quantifier-free. But clearly for any quantifier-free sentence of L, there are models of L where φ is not equivalent to that sentence. Thus, φ cannot be put in prenex normal form.

2.3.8*. Let L be a first-order language. An L-structure A is said to be locally finite if every finitely generated substructure of A is finite. (a) Show that there is a set Ω of quantifier-free types such that for every L-structure A, A is locally finite if and only if A omits every type in Ω. (b) Show that if L has finite signature, then we can choose the set Ω in (a) to consist of a single type.

(a) Ω consists of all types $p(\bar{x})$, where $\bar{x}$ is an n-tuple and p contains formulas of the form $\bigwedge_{i<j<n} t_i(\bar{x}) \neq t_j(\bar{x})$ for t_i, t_j terms of L and arbitrarily large n.

(b) This statement is plainly incorrect, since we must have an n-type for each n, so assume it is one type per n-tuple. Let $S_i(\bar{x})$ be the set of all terms of x with complexity level at most i. Note that for each i, S_i is finite. We let $p(\bar{x})$ be as follows. p contains formulas of the form $\bigvee_{t \in S_n \setminus S_{n-1}} \bigwedge_{s \in S_{n-1}} s(\bar{x}) \neq t(\bar{x})$, for $n < \omega$.

2.4.1. Let L be a first-order language. (a) Suppose ϕ is an $\forall_1$ sentence of L and A is an L-structures. Show that $A \models \phi$ if and only if $B \models \phi$ for every finitely generated substructure B of A. (b) Show that if A and B are L-structures, and every finitely generated substructure of A is embeddable in B, then every $\forall_1$ sentence of L which is true in B is true in A also.

(a) One direction is by Corollary 2.4.2.(a). In the other, suppose that $A \models \neg\phi$. Letting ϕ be $\forall \bar{x} \theta(\bar{x})$ with θ quantifier-free, this means that for some tuple $\bar{a}$ in A, $A \models \neg\theta(\bar{a})$. Then let $B = \langle \bar{a} \rangle_A$, so $B \models \neg\theta(\bar{a})$, so $B \models \neg\phi$.

(b) Let ϕ be any $\forall_1$ sentence. Suppose $A \models \neg\phi$. Then by (a) we can find C a finitely generated substructure of A such that $C \models \neg\phi$. Thus, using the above notation, $C \models \neg\theta(\bar{a})$. We can embed C in B, say by f, so $fC \models \neg\theta(f\bar{a})$. Then $B \models \neg\theta(f\bar{a})$, since θ is quantifier-free, so $B \models \neg\phi$.

2.4.2. Suppose the first-order language L has just finitely many relation symbols and constants, and no function symbols. Show that if A and B are L-structures such that every $\forall_1$ sentence of L which is true in B is true in A too, then every finitely generated substructure of A is embeddable in B.

Let C be any finitely generated substructure of A. Then C is finite. $\mathrm{diag}(C)$ is a quantifier-free sentence in $L(A)$, since there are no terms. Let it be $\psi(\bar{c})$. Then $A \models \exists \bar{x} \psi(\bar{x})$, and so B does too. Letting $\bar{b}$ be the realization in B of $\bar{x}$, C embeds onto $\bar{b}$.

2.4.3. Let L be a first-order language and T a theory in L, such that every $\forall_1$ formula $\phi(\bar{x})$ of L is equivalent modulo T to an $\exists_1$ formula $\psi(\bar{x})$. Show that every formula $\phi(\bar{x})$ of L is equivalent modulo T to an $\exists_1$ formula $\psi(\bar{x})$.

We can assume ϕ is in prenex normal form, say $Q_1\bar{x}_1 \dots Q_n\bar{x}_n\theta(\bar{x}_1, \dots, \bar{x}_n)$, with each Q_i either $\forall$ or $\exists$ (alternating), and θ quantifier-free. Go by induction on n. If Q_n is $\forall$, then $Q_n x_n \theta(\bar{x}_1, \dots, \bar{x}_n)$ is equivalent to an $\exists_1$ formula, by assumption. Then $\exists \bar{x}_{n-1} \forall \bar{x}_n \theta(\bar{x}_1, \dots, \bar{x}_n)$ is equivalent to $\exists \bar{x}_{n-1} \exists \bar{y} \theta^*(\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y})$, which is equivalent to $\exists \bar{x}_{n-1} \bar{y} \theta^*(\bar{x}_1, \dots, \bar{x}_n, \bar{y})$ for some quantifier-free θ^*. Then ϕ is equivalent to a formula with $n-1$ quantifiers, and thus equivalent to an $\exists_1$ formula by induction. If Q_n is $\exists$, then consider $\neg\exists\bar{x}_n(\theta(\bar{x}_1, \dots, \bar{x}_n)) = \forall \bar{x}_n \neg\theta(\bar{x}_1, \dots, \bar{x}_n)$. Since this is $\forall_1$, it is equivalent to $\exists \bar{y} \neg\theta^*(\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y})$, for some quantifier-free θ^*. Thus its negation is equivalent to $\forall \bar{y} \theta^*(\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y})$. Then $\forall \bar{x}_{n-1} \exists \bar{x} \theta(\bar{x}_1, \dots, \bar{x}_n)$ is equivalent to $\forall \bar{x}_{n-1} \bar{y} \theta^*(\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y})$, and so ϕ is equivalent to a formula with $n-1$ quantifiers, and thus equivalent to an $\exists_1$ formula by induction.

2.4.4. In section 2.2 above there are axiomatisations of several important classes of structure. Show that, using the signatures given in section 2.2, it is not possible to write down sets of axioms of the following forms: (a) a set of $\exists_1$ axioms for the class of groups. (b) a set of $\forall_1$ axioms for the class of atomless boolean algebras; (c) a single $\exists_2$ first-order axiom for the class of dense linear orderings without endpoints.

(a) Since $\exists_1$ sentences are preserved in embeddings, take any group G and adjoin an element 0 with the property that $g \cdot 0 = 0$ for all $g \in G$. Then 0 has no inverse, so this structure is not a group.

(b) Since $\forall_1$ sentences are preserved in substructures, take the substructure of just 1 and 0 of any atomless boolean algebra. Then this structure is not an atomless boolean algebra, since 1 is an atom.

(c) Let $\exists\bar{x}\forall\bar{y}\theta(\bar{x},\bar{y})$ be an axiom. Let A be any dense linear order without endpoints. Then we can find $\bar{a}$ in A such that $A \models \forall\bar{y}\theta(\bar{a},\bar{y})$. Since $\forall_1$ formulas are preserved in substructures, take the substructure $B = \langle\bar{a}\rangle_A$, which is just $\bar{a}$. Then $B \models \forall\bar{y}\theta(\bar{x},\bar{y})$, but B is not a dense linear ordering without endpoints.

2.4.5. Let L be as above [with a 2-ary relation symbol $<$]. A Π^0_0 formula is one in which all quantifiers are bounded. A Σ^0_1 formula is a formula in the smallest class of formulas which contains the Π^0_0 formulas and is closed under $\bigwedge$, $\bigvee$ and existential quantification. Show that end-embeddings preserve Σ^0_1 formulas.

Let φ be a Σ^0_1 formula, $A \subseteq B$ an end-embedding, and suppose $A \models \varphi(\bar{a})$. Go by induction on the complexity of φ. Quantifier-free formulas pose no difficulty, neither do negations, conjunctions or disjunctions. We are left with the bounded universal and existential quantifiers. First assume that φ is of the form $\exists x\theta(x,\bar{y})$, for θ Σ^0_0, and θ preserved under end-embeddings. Thus, $A \models \theta(c,\bar{a})$, for some $c \in A$, so $B \models \theta(c,\bar{a})$. Now assume that φ is $\forall x < t(\bar{y})\,\theta(x,\bar{y})$, with θ Π^0_0 and t some term of L. Suppose $B \models \neg\varphi(\bar{a})$. Then $B \models \exists x < t(\bar{a})\,\neg\theta(x,\bar{a})$. But $t(\bar{a})$ is in A, so the witness for x must be in A, so $B \models \neg\theta(\bar{a},c)$, for some $c \in A$. But $A \models \theta(\bar{a},c)$, since $c < t(\bar{a})$ and $A \models \forall x < t(\bar{a})\,\theta(\bar{a},x)$. Contradiction by induction.

2.4.6. Let L be a signature containing a 1-ary symbol P. By a P-embedding we mean an embedding $e : A \to B$, where A and B are L-structures, such that e maps P^A onto P^B. Let Φ be the smallest class of formulas of $L_{\infty\omega}$ such that (i) every quantifier-free formula is in Φ, (ii) Φ is closed under $\bigwedge$ and $\bigvee$, (iii) if ϕ is in Φ and x is a variable then $\exists x\phi$ and $\forall x(Px \to \phi)$ are in Φ. Show that every P-embedding preserves ll the formulas in Φ.

Go by induction on complexity. Quantifier-free formulas, conjunctions, and disjunctions pose no problems. If we have $\exists x\phi(x,\bar{y})$, and $A \models \exists x\phi(x,\bar{a})$, then $A \models \phi(c,\bar{a})$ for some $c \in A$, so $B \models \phi(c,\bar{a})$ by induction, and so $B \models \exists x\phi(\bar{a})$. If we have $\forall x(Px \to \phi)$, and $A \models \forall x(Px \to \phi(x,\bar{a}))$, let b be any element of P^B. Let c be the preimage of b under e. Then $A \models \phi(c,\bar{a})$, so by inductions, $B \models (b,\bar{a})$. Thus, $B \models \forall x(Px \to \phi)$.

2.4.7. Show that if $(A_i \mid i < \gamma)$ is a descending chain of L-structures, then there is a unique L-structure B which is a substructure of each A_j and has domain $\bigcap_{i<\gamma} \operatorname{dom}(A_i)$.

We must show that any L-structures B and B' with domain $\bigcap_{i<\gamma} \operatorname{dom}(A_i)$ which are substructures of each A_i are equal. But in B (and B'), $f(\bar{a}) = b$ iff in all A_i, $f(\bar{a}) = b$, and likewise for relations and constants. Thus, B and B' have the same universe and the same interpretations of functions, relations, and constants. They are then equal.

2.4.8. (a) Show that, if ϕ is a formula of $L_{\infty\omega}$ of the form $\forall \bar{x} \exists_{=n} y \psi(\bar{x}, y, \bar{z})$, where ψ is quantifier-free, then ϕ is preserved in intersections of descending chains of L-structures. (b) Write a set of first-order axioms of this form for the class of real-closed fields. (c) Can axioms of this form be found for the class of dense linear orderings without endpoints?

(a) Suppose not. Let B be the intersection of the descending chain of $\langle A_i \mid i < \gamma \rangle$. Then for some $\bar{a}, \bar{b}$ in B, we have $\neg \exists_{=n} y \psi\left(\bar{a}, y, \bar{b}\right)$, but $A_i \models \forall \bar{x} \exists_{=n} y \psi\left(\bar{x}, y, \bar{b}\right)$. Then each A_i has exactly n witnesses for $\exists_{=n} \psi\left(\bar{a}, y, \bar{b}\right)$, which must then be all the same. Let them be $c_1, \ldots, c_n$. Then $A_i \models \psi\left(\bar{a}, c_j, \bar{b}\right)$, for all $i < \gamma$, $j \leq n$. Since quantifier-free formulas hold in substructures, $B \models \psi\left(\bar{a}, c_j, \bar{b}\right)$, so B must have an additional witness, d. Then $B \models \psi\left(\bar{a}, d, \bar{b}\right)$, but then $A_i \models \psi\left(\bar{a}, d, \bar{b}\right)$, which is impossible.

(b) Modify the given theory of real-closed fields to say $\forall x \exists_{=2} y \left(x = y^2 \vee -x = y^2\right)$. As well, through a great deal of work, it can be shown that the set $\{(x_0, \ldots, x_{n-1}) \mid y^n + x_{n-1} y^{n-1} + \ldots + x_0 \text{ has } e \text{ zeros}\}$ is quantifier-free definable from $(x_0, \ldots, x_{n-1})$. Then, letting $Z_e(x_0, \ldots, x_{n-1})$ be the formula expressing this, we have

$$\forall x_0 \ldots x_{n-1} \exists_{=n} y \left(\bigvee_{0<i\leq n} Z_i(x_0, \ldots, x_{n-1}) \rightarrow \left(y^n + x_{n-1} y^{n-1} + \ldots + x_0 = 0 \vee \bigvee_{j<n-i} y = j + \sum_{k<n} (j + 1 + x_k)^2 \right) \right),$$

saying that if there are i zeros, then y is either a zero or one of $n - i$ fixed reals which cannot be zeros of the polynomial because they are too big.

(c) No, because given any existential witness which is not equal to any other element in the formula, we can find infinitely many such witnesses. Thus, in dense linear orderings, every such statement must have ψ containing a statement of the form $x_i = y$ or $w_i = y$, so we have $\exists_{=1} y$ and in fact y can be eliminated, and the statement rewritten as $\forall_1$, and we know dense linear orderings are not preserved under substructures.

2.4.9. Let L be a signature and Φ the smallest class of formulas of $L_{\infty\omega}$ such that (1) all literals of L are in Φ, (2) Φ is closed under $\bigwedge$ and $\bigvee$, and (3) if $\phi(x, \bar{y})$ is any formula in Φ, then so are the formulas $\forall x \phi$ and $\exists z \forall x (z < x \rightarrow \phi)$. Show that every formula in Φ is preserved under cofinal substructures.

Go by induction on complexity. Literals, conjunctions, and disjunctions pose no problems. As well, $\forall$ has an easy argument, since we are taking substructures. It remains to show that if $A \models \exists z \forall x (z < x \rightarrow \phi(x, \bar{a}))$, then $B \models \exists z \forall x (z < x \rightarrow \phi(x, \bar{a}))$, when B is a cofinal substructure containing $\bar{a}$. Let c be the witness for z in A. Since B is cofinal, we can find d in B with $c < d$. Then certainly $A \models \forall x (d < x \rightarrow \phi(x, \bar{a}))$. Then for every element, b of B with $d < b$, $A \models \phi(b, \bar{a})$. Thus, by induction, $B \models \phi(b, \bar{a})$ as well. Thus, $B \models \forall x (d < x \rightarrow \phi(x, \bar{a}))$, so $B \models \exists z \forall x (z < x \rightarrow \phi(x, \bar{a}))$.

2.4.10. Let L be a signature and L^+ the signature got by adding to L a new n-ary relation symbol P. Let $\bar{x}$ be an n-tuple of variables and $\phi(\bar{x})$ a formula of L^+ in which P is positive. Let A be an L-structure; suppose X and Y are n-ary relations on $\operatorname{dom}(A)$ with $X \subseteq Y$. It is clear that the identity map on A forms an embedding $e : (A, X) \to (A, Y)$ of L^+-structures. (a) Show that e preserves ϕ. (b) For any n-ary relation X on $\operatorname{dom}(A)$ we define $\pi(X)$ to be the relation $\{\bar{a} \mid (A, X) \models \phi(\bar{a})\}$. Show that if $X \subseteq Y$ then $\pi(X) \subseteq \pi(Y)$.

(a) Since e is not in fact an embedding, but a homomorphism, I take it as such. We go by induction on the complexity of ϕ. All literals not mentioning P are obviously preserved, and so are the atomic formulas with P, since if $X(t(\bar{a}))$, for some $\bar{a}$ in A, then $Y(t(e\bar{a}))$, since $X \subseteq Y$. Conjunctions and disjunctions harm nothing. Existential quantification is also easy, and since e is surjective, universal quantification also follows.

(b) e defines an injective mapping from $\pi(X)$ to $\pi(Y)$.

2.5.1. Let L be a first-order language and B an L-structure. Suppose X is a set of elements of B such that, for every formula $\psi(\bar{x}, y)$ of L and all tuples $\bar{a}$ of elements of X, if $B \models \exists y \psi(\bar{a}, y)$ then $B \models \psi(\bar{a}, d)$ for some element d in X. Show that X is the domain of an elementary substructure of B.

We need only show that X is a structure. But B satisfies the formulas $\exists x(x = c)$ and $\exists y(f(\bar{x}) = y)$ for all constants and function symbols, so X contains all constants and is closed under functions, and is thus a substructure.

2.5.2. Give an example of a structure A with a substructure B such that $A \cong B$ but B is not an elementary substructure of A.

Let $B = (\omega, <)$ and $A = (1 + \omega, <)$.

2.5.3. Let B be an L-structure and A a substructure with the following property: if $\bar{a}$ is any tuple of elements of A and b is an element of B, then there is an automorphism f of B such that $f\bar{a} = \bar{a}$ and $fb \in \operatorname{dom}(A)$. Show that if $\phi(\bar{x})$ is any formula of $L_{\infty\omega}$ and $\bar{a}$ a tuple in A, then $A \models \phi(\bar{a}) \Leftrightarrow B \models \phi(\bar{a})$.

We show that $A \preccurlyeq B$. Suppose $B \models \exists y \psi(y, \bar{a})$. Let b be a witness. Then we can take an automorphism of B, f, such that $fb \in A$ and $f\bar{a} = \bar{a}$. But then, since $B \models \psi(fb, f\bar{a})$, as f is an automorphism, $B \models \psi(fb, \bar{a})$, so A and B fulfill the Tarski-Vaught criterion.

2.5.4. Suppose B is a vector space and A is a subspace of infinite dimension. Show that $A \preccurlyeq B$.

Use the previous problem, noting that if we have $B \models \phi(b, \bar{a})$ and $b \notin A$, there is an automorphism sending b to any element in A not in the span of $\bar{a}$ and fixing $\bar{a}$.

2.5.5. Let L be a countable first-order language and B an L-structure of infinite cardinality μ. Show that for every infinite cardinal $\lambda < \mu$, B has an elementary substructure of cardinality λ.

Choose any $X \subseteq B$ of cardinality λ. Form a chain of length λ of increasing sets as follows: enumerate all formulas $\exists y \psi(x, \bar{a})$, with $\bar{a}$ a tuple in X. For each such formula, if B satisfies it, adjoin a witness to X. There are λ such formulas, since L is countable, so after λ-many stages, we have witnesses to every such formula, in our new set X_1, still of cardinality λ. Now repeat. After ω repetitions, we are still of cardinality λ, and now every tuple comes from some finite X_i, and thus every formula on $\bar{a}$ with a realization in B has a realization in X_{i+1}. Now apply Exercise 1.

2.5.6. Let $(A_i \mid i < \gamma)$ be a chain of structures such taht for all $i < j < \gamma$, A_i is a pure substructure of A_j. Show that each structure A_j $(j < \gamma)$ is a pure substructure of the union $\bigcup_{i<\gamma} A_i$.

Let $\varphi(\bar{a})$ be any *p.p.* formula, of the form $\exists \bar{x}(\psi_1(\bar{x}, \bar{a}) \wedge \ldots \wedge \psi_k(\bar{x}, \bar{a}))$ with the union, denoted B, satisfies $\varphi(\bar{a})$. Let A_i be any model containing $\bar{a}$. We must show that $A_i \models \varphi(\bar{a})$. The witnesses for $\bar{x}$ in B have come at some finite stage, say A_j. If $j < i$, we are done, since the ψ_k's are atomic, $B \models \varphi(\bar{a}) \Rightarrow A_j \models \varphi(\bar{a}) \Rightarrow A_i \models \varphi(\bar{a})$. If $j > i$, then since A_j is a pure extension of A_i and $A_j \models \varphi(\bar{a})$, $A_i \models \varphi(\bar{a})$.

2.5.7. Show that if n is a positive integer, then there are a first-order language L and L-structures A and B such that $A \subseteq B$ and for every n-tuple $\bar{a}$ in A and every formula $\phi(x_0, \ldots, x_{n-1})$ of l, $A \models \phi(\bar{a}) \Leftrightarrow B \models \phi(\bar{a})$, but B is not an elementary extension of A.

Let L be the language of a $(2n+2)$-ary relation symbol R, a $(2n+2)$-ary equivalence relation E, and an $n+1$-ary relation P. Let $\mathbf{K}$ be the class of all finite structures with the properties that E is an equivalence relation on *sets* of size $n+1$, R defines a linear order on the equivalence classes of E, and P is true on at most one equivalence class, which has no elements less than it in the R-ordering. We can ensure that E is defined on sets by having all $n+1$-tuples with a repeated element in the same equivalence class, which is not ordered by R. $\mathbf{K}$ clearly has HP, JEP, and AP. Let C^+ be its Fraïssé limit. Let C be C^+ without P in the language. Let $\{a_0, \ldots, a_n\}$ be a set in what was P. Let $\{b_0, \ldots, b_n\}$ be a set disjoint from the a_i's which was not in P. Since the a_i's and the b_i's have the same quantifier-free type, it is not hard to see, through a back-and-forth argument, that there is an isomorphism of C taking a_i to b_i, f. Let the image of this isomorphism be $A \subset C$. Then necessarily A contains no $n+1$-tuples which are in any equivalence class less than that of $\{b_0, \ldots, b_n\}$. Now, let $\phi(\bar{a})$ be any formula with at most n parameters from A. Clearly, $C \models \phi(\bar{a}) \Leftrightarrow A \models \phi(f\bar{a})$. I show that, for $\bar{a}$ of length at most n, $C \models \phi(\bar{a}) \Leftrightarrow C \models \phi(f\bar{a})$, which is sufficient. But this follows by the ultrahomogeneity of C^+, as in C^+, these two tuples have the same quantifier-free type (since P cannot be true on any term of them), and thus there is an automorphism of C^+, which is an automorphism of C, taking one to the other. But clearly, if $\bar{a}$ is an $n+1$-tuple, C and A can disagree, say on the formula $\exists x_0 \ldots x_{n-1}((x_0, \ldots, x_{n-1}) R\bar{a})$.

2.5.8. Suppose R is a ring and A, B are left R-modules with $A \subseteq B$, and, for every element a and A and every p.p. formula $\phi(x)$ without parameters, $A \models \phi(a) \Leftrightarrow B \models \phi(a)$. Show that A is pure in B.

Go by induction on n. Let $\bar{a}$ be an n-tuple in A with $B \models \phi(\bar{a})$. Then $B \models \exists x_n \phi(\bar{a}|n, x_n)$. By induction hypothesis, there is then a c in A such that $A \models \phi(\bar{a}|n, c)$. Thus, $B \models \phi(\bar{a}|n, c)$.

We now examine what this means. $\phi(\bar{y})$ is of the form $\exists\bar{x}(\psi_1(\bar{x},\bar{y})\wedge\ldots\wedge\psi_n(\bar{x},\bar{y}))$, with each ψ_i an equation, say of the form $\sum_{j<k}p_j^i x_i+\sum_{j<n+1}q_j^i y_i=0$, with p_j^i and q_j^i elements of R. Since we have $\phi(\bar{a})$, this gives us $\exists\bar{x}\bigwedge_{i\leq n}\sum_{j<k}p_j^i x_i+\sum_{j<n+1}q_j^i a_i=0$. Let $\bar{b}$ witness $\bar{x}$, so we have $\bigwedge_{i\leq n}\sum_{j<k}p_j^i b_i+\sum_{j<n+1}q_j^i a_i=0$. As well, we have $\phi(\bar{a}|n,c)$, so by the same argument, we have $\bar{d}$ with $\bigwedge_{i\leq n}\sum_{j<k}p_j^i d_i+\sum_{j<n}q_j^i a_i+q_{n+1}^i c=0$. Now, for each i, by setting the two left sides equal to each other (since they are both 0), and then gathering terms, we have $\sum_{j<k}p_j^i(b_i-d_i)+q_{n+1}^i(a_n-c)=0$, and thus $\exists\bar{x}(\psi_1(\bar{x},(0,\ldots,0,a_n-c))\wedge\ldots\wedge\psi_n(\bar{x},(0,\ldots,0,a_n-c)))$, so $B\models\phi(0,\ldots,0,a_n-c)$. But this expression can easily be rewritten so that there are fewer than $n+1$ parameters – in fact, 1 will do. Thus, by induction, $A\models\phi(0,\ldots,0,a_n-c)$. Performing the same procedure as above, we can add this solution to $(\bar{a}|n,c)$ to yield $A\models\phi(\bar{a})$.

2.5.9. Let R be a ring and L the language of left R-modules. Let M be a left R-modul and $\phi(x_0,\ldots,x_{n-1})$ a p.p. formula of L. (a) Show that $\phi(M^n)$ is a subgroup of M^n regarded as an abelian group. (b) Show that if $\phi(\bar{x},\bar{y})$ is a p.p. formula of L and $\bar{b}$ a tuple from M, then $\phi\left(M^n,\bar{b}\right)$ is either empty or a coset of the p.p.-definable subgroup of $\phi(M^n,0,\ldots,0)$. Show that both possibilities can occur.

(a) The above argument shows that $\phi(M^n)$ is closed under addition, and multiplying through by -1 shows that it is closed under inverses as well. Thus it is an abelian group.

(b) If $\phi\left(M^n,\bar{b}\right)$ is not empty, we can perform the above procedure to show that the difference of any two elements of $\phi\left(M^n,\bar{b}\right)$ is an element of $\phi(M^n,0,\ldots,0)$, and that given any element of $\phi\left(M^n,\bar{b}\right)$, adding any element of $\phi(M^n,0,\ldots,0)$ preserves its membership. Thus, it is a coset. If we let $\psi(x,y_1,y_2)$ be $0x+0y_1+y_2=0$, and $b\neq 0$, the set $\exists x\psi(x,y_1,b)$ is empty. If we let $\psi(x,y_1,y_2)$ be $0x+y_1+y_2=0$, then whatever b is, there is a unique solution which is a coset of $\{0\}$.

2.6.1. Let L be a signature. Form a signature L^r from L as follows: for each positive n and each n-ary function symbol F of L, introduce an $(n+1)$-ary relation symbol R_F. If A is an L-structure, let A^r be the L^r-structure got from A by interpreting each R_F as the relation $\{(\bar{a},b)\mid A\models(F\bar{a}=b)\}$. (a) Define a translation $\phi\to\phi^r$ from formulas of L to formulas of L^r, which is independent of A. Formulate and prove a theorem about this translation and the structures A, A^r. (b) Extend (a) so as to translate every formula of L into a formula which contains no function symbols and no constants.

(a) Given ϕ, we can assume that ϕ is unnested. Now, replace each atomic formula of the form $F(\bar{x})=y$ with $R_F(\bar{x}y)$. This is ϕ^r. The theorem is that $A\models\phi(\bar{a})\Leftrightarrow A^r\models\phi^r(\bar{a})$ for any tuple $\bar{a}$ in A. It is clearly true for atomic formulas, and so by induction.

(b) For each constant symbol c, introduce a relation R_c, interpreted on A^r as $\{a\mid A\models a=c\}$.

2.6.2. Show that if L is a first-order language, then every formula $\phi(\bar{x})$ of L is logically equivalent to a negation normal formula $\phi^*(\bar{x})$ of L.

We go by induction on the complexity of formulas. For atomic and negations of atomic formulas, there is nothing to do. The cases $\phi = \bigwedge \psi_i$, $\phi = \bigvee \psi_i$, $\phi = \exists x \psi$, and $\phi = \forall x \psi$ are done by induction, since if the ψs are in negation normal form, so is ϕ. Suppose $\phi = \neg \varphi$. Now we go by induction on complexity of φ. Go through the above cases. If $\varphi = \bigwedge \psi_i$, then $\neg\varphi$ is equivalent to $\neg \bigwedge \psi_i$, so $\bigvee \neg \psi_i$, and we are done by induction. Similarly for $\bigvee$. $\exists x \psi$ yields $\forall x \neg \psi$, and similarly for $\forall$. Finally, if φ is $\neg\psi$, then ϕ is ψ, and by induction we are done.

2.6.3. Let L be a first-order language, R a relation symbol of L and ϕ a formula of l Show that the following are equivalent. (a) R is positive in some formula of L which is logically equivalent to ϕ. (b) ϕ is logically equivalent to a formula of L in negation normal form in which R never has $\neg$ immediately before it.

Since every negation normal formula as above is in the class of formulas which are positive in R (since negation normal formulas as above can include any literal positive in R, and are closed under $\bigwedge$, $\bigvee$, and quantification), (b) implies (a). Conversely, it is easy to see that any formula which is positive in R is in the required negation normal form.

2.6.4. Let T be the theory of linear orderings. For each positive integer n, write a first-order sentence which expresses (modulo T) 'There are at least n elements', and which uses only two variables, x and y.

$\exists x \left(\exists y \left(y < x \wedge \exists x \left(x < y \wedge \exists y \left(\ldots\right)\right)\right)\right)$.

2.6.5. Let T_0, T_1 and T_2 be first-order theories. Show that if T_2 is a definitional expansion of T_1 and T_1 is a definitional expansion of T_0, then T_2 is a definitional expansion of T_0.

We must show that, for every symbol of $L_2 \setminus L_0$, there is a definition of it in T_2 in terms of L_0. Let R be any such symbol. Since T_2 is a definitional expansion of T_1, $T_2 \vdash \forall x \left(R\left(\bar{x}\right) \leftrightarrow \theta\left(\bar{x}\right)\right)$, for some θ in L_1. Since T_1 is a definitional expansion of T_0, for any symbol P in $L_1 \setminus L_0$, $T_1 \vdash \forall \bar{x} \left(P\left(\bar{x}\right) \leftrightarrow \psi\left(\bar{x}\right)\right)$, for some formula ψ in L_0. Thus, T_1 proves the equivalence of any formula in L_1 with one in L_0, and thus T_2 does too, since it is a definitional expansion. Thus, we have θ^* in L_0 equivalent to θ modulo T_2, so $T_2 \vdash \forall \bar{x} \left(R\left(\bar{x}\right) \leftrightarrow \theta^*\left(\bar{x}\right)\right)$, and so T_2 is a definitional expansion of T_0. (The other parts of being a definitional expansion are trivially checked.)

2.6.6. Let L an L^+ be signatures with $L \subseteq L^+$; let T be a theory of signature L^+ and S a symbol of signature L^+. Suppose that there are two models A, B of T such that $A|L = B|L$ but $S^A \neq S^B$. Deduce that A is not explicitly definable in T in terms of L.

We show the contrapositive. Suppose S is explicitly defined by T. Then $T \vdash \forall \bar{x} \left(S\left(\bar{x}\right) \leftrightarrow \theta\left(\bar{x}\right)\right)$, say. Then, since $A \models T$ and $B \models T$, $A \models \forall \bar{x} \left(S\left(\bar{x}\right) \leftrightarrow \theta\left(\bar{x}\right)\right)$, and likewise for B. Now, since $S^A \neq S^B$, we can find $\bar{a}$ in $S^A \setminus S^B$ (WLOG). Then $A \models \theta\left(\bar{a}\right)$, but $B \models \neg\theta\left(\bar{a}\right)$. But since θ is in L, then $A|L \neq B|L$.

2.6.7. Let L^+ be the first-order language of arithmetic with symbols $0, 1, +, \cdot$, and let T be the complete theory of the natural numbers in this language. Let L be the language L^+ with the symbol $+$ removed. Show that $+$ is not explicitly definable in T in terms of L.

Without $+$, $\mathbb{N}$ is isomorphic to a free abelian group on countably many generators (the primes), along with an element 0. We can thus define $+$ on this structure either as usual, or by first taking an isomorphism between 2 and 3 and then defining it as usual. Let A be the first model, and B the second. Then we have $A|L = B|L$, but $2 +^A 2 = 4$, while $2 +^B 2 = 6$.

2.6.8. (a) Show that if $T \subseteq T^+$ and every L-structure which is a model of T can be expanded to form a model of T^+, then T^+ is a conservative extension of T. In particular every definitional expansion is conservative. (b) Prove that the converse of (a) fails.

(a) Suppose T^+ is not a conservative extension of T. Then, for some ϕ, either $T^+ \vdash \phi$ and $T \vdash \neg\phi$, or $T^+ \vdash \phi$ and $T \nvdash \phi$. The first case is trivial. In the second, take a model $A \models T$ with $A \models \neg\phi$, possible since T does not prove ϕ. Then plainly A cannot be expanded to a model of T^+.

(b) Let T be the theory of $(\omega, <)$. Let T^+ be Peano arithmetic. Clearly, T^+ and T have the same L-consequences. However, T has a model $\omega + \mathbb{Z}$. This cannot be a model of T^+: let $(1, 0)$ denote the 0 element of $\mathbb{Z}$. If we can expand this model to a model of T^+, there must be some a with $a + a = (1, 0)$. But if $a \in \omega$, then $a + a < (1, 0)$, and if $a \in \mathbb{Z}$, then let $a = (1, -n)$, for some $n \in \mathbb{N}$. Then $(1, 0) - (1, -n) = n$, and so, since $a > n$, $a + a > a + n = (1, 0)$. Thus, a does not exist, and so this cannot be a model of T^+.

2.6.9. Let L be the language with constant symbol 0 and 1-ary function symbol S; let L^+ be L with a 2-ary function symbol $+$ added. Let T^+ be the theory $\forall x x + 0 = x$, $\forall xy x + Sy = S(x + y)$. Show that T^+ is not a conservative extension of the empty theory in L.

We must show that there is an L-structure, A, such that A cannot be expanded to a model of T. Let A have three elements, 0, 1, and $1'$, with $S0 = 1$, $S1 = 0$, and $S1' = 0$. Now, try to make A a model of T. We must define $1' + 1'$. Suppose $1' + 1' = 0$. Then $1' + S1' = S(1' + 1') = S0 = 1$, so $1' = 1$, which is false.

2.6.10. Show that the theory of boolean algebras is definitionally equivalent to the theory of commutative rings with $\forall x x^2 = x$.

We translate $\cap$ as $\cdot$ and $\cup$ as $+$. x^* is translated as $1 - x$. Commutativity, associativity, and distributivity are the same for both. The existence of a multiplicative identity is an axiom of commutative rings and easily proved for boolean algebras. The existence of an additive identity is the same. Finally, boolean algebras have $\forall x x \cap x = x$, equivalent to a commutative ring's condition that $x^2 = x$, and boolean algebras have $x \cap x^* = 0$, which is true in such commutative rings, since $x(1 - x) = x - x^2 = 0$.

3.1.1. Let L be a first-order language and L' a language which comes from L by adding constants.

Show that if T is a Skolem theory in L, then T is a Skolem theory in L' too (and hence so is any theory $T' \supseteq T$ in L').

Consider any formula $\phi(\bar{x}, y)$ of L'. Let $\bar{c}$ be the new constants of L' which are used in ϕ. Then write ϕ as $\phi'(\bar{x}, \bar{c}, y)$, with $\phi'(\bar{x}, \bar{z}, y)$, a formula of L. Since T is a Skolem theory, $T \vdash \forall \bar{x}\bar{z}(\exists y \phi'(\bar{x}, \bar{z}, y) \rightarrow \phi(\bar{x}, \bar{z}, t(\bar{x}, \bar{z})))$, for some term t of L. But then, by the lemma on constants, $T \vdash \forall \bar{x}(\exists y \phi'(\bar{x}, \bar{c}, y) \rightarrow \phi'(\bar{x}, \bar{c}, t(\bar{x}, \bar{c})))$, so $T \vdash \forall \bar{x}(\exists y \phi(\bar{x}, y) \rightarrow \phi(\bar{x}, t'(\bar{x})))$, with $t'(\bar{x}) = t(\bar{x}, \bar{c})$ a term of L'. Thus, T remains Skolem.

3.1.2. Use the downward Löwenheim-Skolem theorem and the result of Example 3 in the next section to show that if A and B are dense linear orderings without endpoints, then $A \equiv B$.

Let A and B be any two dense linear orders without endpoints (DLOWE). By the downward Löwenheim-Skolem theorem, there are countable $A' \preccurlyeq A$ and $B' \preccurlyeq B$. By Example 3, $A' \cong B'$, so $A \equiv A' \equiv B' \equiv B$.

3.1.3. Show that, if T is a first-order theory which has Skolem functions, then T is model-complete. Give an example of a first-order theory which is model-complete but doesn't have Skolem functions.

We show that if $A \models T$, $B \models T$, and $A \subseteq B$,then $A \preccurlyeq B$. By the Tarski-Vaught criterion, it suffices to show that whenever $B \models \exists y \varphi(\bar{a}, y)$, there exists $d \in A$ such that $B \models \varphi(\bar{a}, d)$, for all $\varphi \in L$ and $\bar{a}$ in A. Since T has Skolem functions, $T \vdash \forall \bar{x}(\exists y \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t(\bar{x})))$ for some term t. Thus, $B \models \varphi(\bar{a}, t(\bar{a}))$. But since A is a substructure, $t(\bar{a})$ is the desired d. Thus, $A \preccurlyeq B$.

The theory of DLOWE's is model-complete, but does not have Skolem functions. It is model-complete because it has elimination of quantifiers, so $\exists y \varphi(\bar{a}, y)$ reduces to a statement just about the ordering of $\bar{a}$, which is true in A iff it is true in B. It does not have Skolem functions since it has no non-trivial terms.

3.1.4*. Let L be a first-order language with at least one constant. Show that if T is a Skolem theory in L, then T has elimination of quantifiers.

We know that every formula $\phi(\bar{x})$ with $\bar{x}$ non-empty is equivalent to a quantifier-free formula. The question thus reduces to asking if sentences of the form $\exists y \theta(y)$ and $\forall y \theta(y)$, with θ quantifer-free, are equivalent to quantifier-free sentences in T (i.e., true or false). Clearly, we need only deal with the first kind. Let $\theta'(z, y)$ be $\theta(y) \wedge z = c$. Then certainly $T \vdash \exists x \theta(x) \leftrightarrow \exists z \exists y \theta'(z, y)$. But $\exists y \theta'(z, y)$ has unquantified variables, so it is equivalent, modulo T, to $\theta^*(z)$, with θ^* quantifier-free. Now, if $A \models T$ is a model with $A \models \theta(a)$, for some $a \in A$, then certainly $A \models \exists y \theta'(c, y)$, so $A \models \theta^*(c)$. Conversely, suppose $A \models \theta^*(c)$. Then $A \models \exists y \theta'(c, y)$, so $A \models \exists x \theta(x)$. Thus, modulo T, $\exists x \theta(x)$ and $\theta^*(c)$ are equivalent. Thus, T has elimination of quantifiers.

3.1.5*. Suppose T is a theory in a first-order language L, and, for every quantifier-free formula $\phi(\bar{x}, y)$ of L with $\bar{x}$ non-empty, there is a term t of L such that T entails the sentence $\forall \bar{x}(\exists y \phi(\bar{x}, y) \rightarrow \phi(\bar{x}, t(\bar{x})))$.

(a) Show that T has Skolem functions. (b) Show that for any theory T' in L with $T \subseteq T'$, T' is equivalent to an $\forall_1$ theory.

(a) Lemma 2.3.1 shows that every formula $\phi(\bar{x}, y)$ with $\bar{x}$ non-empty is equivalent to a quantifier-free formula $\phi^*(\bar{x}, y)$. Thus, we can find the appropriate term for $\exists y \phi(\bar{x}, y)$ by looking at ϕ^*.

(b) Note that all of the above sentences are $\forall_1$. Since they collectively imply that every other sentence is either an $\exists_1$ sentence, a $\forall_1$ sentence, or true or false, we know that T' is equivalent to a theory with sentences of the above form, of the form $\exists x \theta(x)$, and $\forall x \theta(x)$, with θ quantifier-free. We must show that the first type is actually equivalent to a $\forall_1$ sentence. But this is false if L has no constants. Let T be exactly those sentences above, and let $T' = T \cup \{\exists x (x = x)\}$. Since every sentence of T is $\forall_1$, the empty model is a model of T. $\exists x (x = x)$ cannot be equivalent to a $\forall_1$ sentence, since then that sentence would be true in the empty model. But the empty model is not a model of T'. If L has a constant symbol, or if the empty model is excluded:

Let $\exists x \theta(x)$ be any sentence in T', with θ quantifier-free. We show that it is equivalent to a $\forall_1$ sentence, modulo T. Let $\theta'(z, y)$ be $\theta(y) \wedge z = z$. Then $\exists x \theta(x) \leftrightarrow \forall z \exists y \theta(y, z)$. Since T' is Skolem, we have $\exists y \theta(y, z) \leftrightarrow \theta^*(z)$, with θ^* quantifier-free. Then $\exists x \theta(x) \leftrightarrow \forall z \theta^*(z)$, so $\exists x \theta(x)$ is equivalent to a $\forall_1$ sentence.

3.1.6. Let L be a first-order language and A an L-structure which has Skolem functions. Suppose X is a set of elements which generate A, and $<$ is a linear ordering of X (not necessarily expressible in L). Show that every element of A has the form $t^A(\bar{c})$ for some term $t(\bar{x})$ of L and some tuple $\bar{c}$ from X which is strictly increasing in the sense of $<$.

Every element of A has the form $t^A(\bar{c})$, for some term $t(x_0, \ldots, x_{n-1})$ and some tuple $\bar{c}$ from X. It is possible that $\bar{c} = (c_0, \ldots, c_{n-1})$ is not increasing in X in the sense of $<$. Let $\bar{d}$ be the tuple where the c_i's are put in increasng order. Let $i(m)$ be the index of c_i in $\bar{d}$. (Thus, if c_5 is the first element of $\bar{d}$, $i(5) = 0$.) Then it is easy to construct a new term $t'(x_0, \ldots, x_{n-1}) = t\left(x_{i(0)}, \ldots, x_{i(n-1)}\right)$ with $t'\left(\bar{d}\right) = t(\bar{c})$, so $t(\bar{c})$ has the desired form.

3.1.7. Let **K** be the class of boolean algebras which are isomorphic to power set algebras of sets. Show that **K** is not first-order axiomatisable.

Let A be isomorphic to the power set algebra of ω. Then A is not countable. However, A has a countable elementary substructure, A'. A has infinitely many atoms, and this fact is in $\mathrm{Th}(A)$, so A' has infinitely many atoms. But then A' cannot have the full power set of its atoms. Since no set of first-order sentences distinguishes A from A' in the language of boolean algebras, **K** cannot be first-order axiomatizable.

3.1.8. Let L be a finite relational signature and A an infinite L-structure. Suppose there is a simple group which acts transitively on A. Show that A has a countable elementary substructure on which some simple group acts transitively.

Let G be the simple group acting transitively on A. Fix any a in A. If g and h are any distinct elements of G then $ga \neq ha$, since otherwise $a = g^{-1}ha$, but $g^{-1}h \neq 1$, and so the subgroup of G $\{g \mid ga = a\}$ is a proper normal subgroup. Thus, each element of G is associated to exactly one element of A – the image of a under that element – and the association is bijective, since G is transitive. Thus, we can regard G as an expansion of A. Then let H be a countable elementary substructure of G as an expansion of A. H is simple by Example 1, and acts transitively on itself. But it can also be interpreted as a countable elementary substructure of A, thus it is the desired one.

3.2.1. Prove Lemma 3.2.1(b) [If $\beta < \gamma$ and $A \sim_\gamma B$ then $A \sim_\beta B$].

Let player $\exists$ follow the strategy for $\mathrm{EF}_\gamma(A, B)$ when playing $\mathrm{EF}_\beta(A, B)$. After β stages, if $(A, \bar{a}) \not\equiv_0 \left(B, \bar{b}\right)$, then certainly after γ stages they will not be 0-equivalent, so since $\exists$ can win the γ game, after β stages they must be 0-equivalent, so $\exists$ wins the β game.

3.2.2. (a) Show that, in the game $\mathrm{EF}_\gamma(A, B)$ a strategy σ for a player can be written as a family $(\sigma_i \mid i < \gamma)$ where for each $i < \gamma$, σ_i is a function which picks the player's i-th choice $\sigma_i(\bar{x})$ as a function of the sequence $\bar{x}$ of previous choices of the two players. (b) Show that σ can also be written as a family $(\sigma'_i \mid i < \gamma)$ where for each $i < \gamma$, σ'_i is a function which picks the player's i-th choice $\sigma'(\bar{y})$ as a function of the sequence $\bar{y}$ of previous choices of the *other* player. (c) How can the functions σ_i be found from the functions σ'_i, and vice versa?

(a) We assume the domains of A and B are disjoint, for simplicity. We define σ_i. Let $\left(\bar{c}, \bar{d}\right)$ be the choices made so far (lengths are equal if the strategy is for $\forall$, and one is one element longer if the strategy is for $\exists$). Since we have a strategy σ, there is a rule for choosing the next element; in other words, the next element must satisfy some definable properties, and such an element always exists in $A \cup B$. Then, by choice, we can take $\sigma_i\left(\bar{c}, \bar{d}\right)$ to be a specific such element. Doing this for all possible choices of $\left(\bar{c}, \bar{d}\right)$ of the appropriate lengths defines σ_i, and such elements can always be found because σ is a winning strategy. It goes without saying that $\sigma_i\left(\bar{c}, \bar{d}\right)$ continues the winning strategy.

(b) We define σ_i, given $(\sigma_j \mid j < i)$ and $\bar{c}$, a list of the opponent's moves, with the assumption that for each $j < i$, σ_j preserves the winning strategy of σ. Then $(\bar{c}, \langle \sigma_j(\bar{c}|j) \mid j < i\rangle)$ is a winning position for the player, and it is definable from $\bar{c}$. Then, by the same choice procedure, we can let σ_i be the next move under the strategy σ.

(c) Finding σ_i from σ'_i is trivial – $\sigma_i\left(\bar{c}, \bar{d}\right) = \sigma'_i(\bar{c})$. The other way, we define inductively. $\sigma'_0 = \sigma_0$. If we have defined σ'_j in terms of σ_j for all $j < i$, let $\sigma'_i(\bar{c}) = \sigma_i\left(\bar{c}, \left\langle \sigma'_j(\bar{c}) \mid j < i\right\rangle\right)$.

3.2.3. The game $P_\omega(A, B)$ is defined exactly like $\mathrm{EF}_\omega(A, B)$ except that player $\forall$ must always choose from structure A and player $\exists$ from structure B. Show that, if A is at most countable, then player $\exists$ has a winning strategy for $P_\omega(A, B)$ if and only if A is embeddable in B. (b) What if player $\forall$ must choose from structure A in even-numbered steps and from structure B in odd-numbered steps (and player $\exists$ vice versa).

Suppose A is embeddable in B. Then if player $\exists$ just follows the embedding map, she will guarantee a win. Conversely, if $\exists$ can win any game, let $\forall$ play to exhaust A, and let his play be $\bar{a}$. Then $(A, \bar{a}) \equiv_0 (B, \bar{b})$, so the diagram of A is in B, thus A embeds into B by taking $\bar{a}$ to $\bar{b}$.

(b) This game (denoted P') is equivalent to the $\mathrm{EF}_\omega(A, B)$ game. Note that P' imposes no restrictions on $\exists$, since her move must always be in the opposite model from $\forall$'s. Thus, if $\exists$ has a winning strategy in $\mathrm{EF}_\omega(A, B)$, she has one in P' too. Suppose $\exists$ has a winning strategy in P'. We translate it to a winning strategy in $\mathrm{EF}_\omega(A, B)$. $\exists$ plays the following side game in P'. If at turn i, $\forall$ chooses from the required model in P', then in $\exists$'s side game, her $\forall$ makes that same choice. Since $\exists$ can win P', she uses her winning strategy there to make her move. If $\forall$ chooses from the incorrect model, $\exists$ first has her $\forall$ player make any choice from the correct model, makes her response to that move based on her strategy, and then has her $\forall$ make the real $\forall$'s choice, which is now in the right model. Then she applies her strategy to find a response. In the end, her side game will have sequences $\bar{a}$ and $\bar{b}$ with $(A, \bar{a}) \equiv_0 (B, \bar{b})$, and the real game will have subsequences, $\bar{a}'$ and $\bar{b}'$, but then certainly $(A, \bar{a}') \equiv_0 (B, \bar{b}')$.

3.2.4. The game $H_\omega(A, B)$ is defined exactly like $\mathrm{EF}_\omega(A, B)$ except that player $\exists$ wins the play $(\bar{a}, \bar{b})$ iff for every atomic formula ϕ, $A \models \phi(\bar{a}) \Rightarrow B \models \phi(\bar{b})$. Show that, if A and B are at most countable, then player $\exists$ has a winning strategy for $H_\omega(A, B)$ if and only if B is a homomorphic image of A.

If B is a homomorphic image of A, $\exists$ can just use the homomorphism as a strategy. Conversely, if $\exists$ has a winning strategy, let $\forall$ play to exhaust A and B. Then $\exists$'s moves define a homomorphism from A onto B.

3.2.5. Suppose A and B are two countable dense linearly ordered sets without endpoints, both partitioned into classes $P_0, \ldots, P_{n-1}$ so that each class P_i occurs densely in both orderings. Show that A is isomorphic to B.

We show A is back-and-forth equivalent to B, which suffices, since they are both countable. Let $\bar{a}$ and $\bar{b}$ from A and B respectively be given, with $(A, \bar{a}) \equiv_0 (B, \bar{b})$. Let $\forall$ choose $c \in A$ (WLOG). Suppose $a_i < c < a_j$, with no a_k between a_i and a_j (the cases $c < a_i$, all i, or $a_i < c$, all i, are handled similarly). Suppose $P_k(c)$. Then, since P_k is dense in B, we can find d, with $b_i < d < b_j$ and $P_k(d)$, so $(A, \bar{a}, c) \equiv_0 (B, \bar{b}, d)$, and so we can extend indefinitely, so $\exists$ has a winning strategy, which is just to preserve 0-equivalence.

3.2.6. (a) If ζ is a linear ordering, let ζ^+ be the ordering which we get by replacing each point of ζ by a pair of points a, b with $a < b$. Show that if ζ and ξ are dense linear orderings without endpoints then ζ^+ and ξ^+ are back-and-forth equivalent. (b) Show that, if ζ and ξ are dense linear orderings which have first points but no last points, then ζ is back-and-forth equivalent to ξ.

(a) For ease of notation, let $P(a, b)$ be true iff a and b are as above in ζ^+ (or ξ^+), and $I(a)$ be true if a is the first element in a pair. Let $\bar{a}$ and $\bar{b}$, from A and B respectively, be such that $(A, \bar{a}) \equiv_0 (B, \bar{b})$, and for each $a_i, a_j \in \bar{a}$, $P(a_i, a_j) \Leftrightarrow I(b_i, b_j)$, and $I(a_i) \Leftrightarrow I(b_i)$. Now let $\forall$ choose any c in A. Let $a_i < c < a_j$ be its place in $\bar{a}$. If $P(a_i, c)$ or $P(c, a_j)$, choose d to be the corresponding element for b_i or b_j. This choice will preserve the desired properties since for any other b_k, if $b_k = d$, then $P(b_i, b_k)$ (for example), so $P(a_i, a_k)$, so $c = a_k$. Inequalities are easily seen to be preserved. If c is not paired with a_i or a_j, let d be any element between b_i and b_j which is not paired with either, and has $I(d) \Leftrightarrow I(c)$. Then d preserves the desired properties. Thus, $\exists$ has a strategy which preserves 0-equivalence, and thus a winning strategy.

(b) We can add tails to ζ and ξ and make them DLOWE's. Then they are back-and-forth equivalent. But any game in the original ζ and ξ corresponds to a game where $\forall$ chooses the first element of ζ, $\exists$ responds with the first element of ξ (which is a winning strategy), and then $\forall$ only chooses elements above those elements after that. Since $\exists$ can win this game, she can win the original back-and-forth game between ζ and ξ.

3.2.7. Let A, B and C be fields; suppose $A \subseteq B$, $A \subseteq C$, and both B and C are algebraically closed and of infinite transcendence degree over A. Let (B, A) be the structure consisting of B with a 1-ary relation symbol P added so as to pick out A; and likewise with (C, A) and C. Show that (B, A) is back-and-forth equivalent to (C, A).

$\exists$'s strategy is as follows. Let $\left(\bar{b}, \bar{c}\right)$ be the plays so far from B and C, with $\left(B, \bar{b}\right) \equiv_0 (C, \bar{c})$. If $\forall$ chooses an element (of B, say) which is algebraic over $A \cup \bar{b}$, let $p\left(x, \bar{b}\right)$ be its minimal polynomial over $A \cup \bar{b}$. Since C is algebraically closed, there is a solution to $p(x, \bar{c})$, since if the polynomial has no solutions, it must be trivial, which is a quantifier-free condition on $\bar{c}$, and therefore also true on $\bar{b}$. Let $\exists$'s response be a solution to this polynomial. (Note that this strategy maps each element of A to itself.) If $\forall$ chooses an element which is transcendental over $A \cup \bar{b}$, $\exists$ chooses any element transcendental over $A \cup \bar{c}$, possible since C (and B) have infinite transcendence degree. Let $\bar{b}$, $\bar{c}$ be the sequences at the end of the game. Suppose $\left(B, \bar{b}\right) \not\equiv_0 (C, \bar{c})$. Then for some atomic formula $\theta(\bar{x})$, we have $B \models \theta(b_{i_1}, \ldots, b_{i_n})$, but $C \models \neg\theta(c_{i_1}, \ldots, c_{i_n})$ (or vice versa). θ must be an equality, so, assuming that $i_n > i_j$ for $j < n$, we have a polynomial expression for b_{i_n} in terms of the other b_{i_j}'s. But then by construction, b_{i_n} and c_{i_n} satisfy the same minimal polynomial, which must then divide the polynomial in θ, so the polynomial with c_{i_j}'s is 0 as well, so $C \models \theta(c_{i_1}, \ldots, c_{i_n})$.

3.2.8. Show that if A, B are respectively back-and-forth equivalent to A', B', then their disjoint sum $A + B$ is back-and-forth equivalent to $A' + B'$.

$\exists$ plays two side games, one on A, A' and one on B, B' when playing the game between $A + B$ and $A' + B'$, yielding sequences $\left(\bar{a}, \bar{b}, \bar{a}', \bar{b}'\right)$, with $(A, \bar{a}) \equiv_0 (A', \bar{a}')$, and the same for B and B'. Note that any atomic formula with elements of $\bar{a}$ and $\bar{b}$ is false in $A + B$ (and similarly for $A' + B'$), so any atomic formula with parameters from $A + B$ is equivalent to two atomic formulas, one with parameters from A and one with parameters from B, and by $\exists$'s strategy, both have the same truth value in A' and B', and hence in $A' + B'$.

3.3.1. Prove Theorem 3.3.2(c) [for every $k < \omega$ and every unnested formula $\phi(\bar{x})$ of L with n free variables $\bar{x}$ and quantifier rank at most k, we can effectively find a disjunction $\theta_0 \vee \ldots \vee \theta_i(\bar{x})$ in $\Theta_{n,r}$ which is logically equivalent to ϕ]. [Note that the definition in the text of $\Theta_{n,k+1}$ is incorrect – for the conjunction, it should be $i \notin X$ and $\neg\theta_i$.]

Go by induction on k. Since $\Theta_{n,0}$ includes all conjunctions of primitives, and ϕ can be written as a disjunction of conjunctions, ϕ is certainly a disjunction in $\Theta_{n,0}$, and since the process of writing ϕ as a disjunction of conjunctions is effective, we can find the appropriate elements of $\Theta_{n,0}$ effectively. Now we do it for $k+1$. Suppose ϕ has quantifier-rank $k+1$. Note that $\Theta_{n,k+1}$ is closed under conjunctions, and certainly if each term in a disjunction is writeable as a disjunction of elements of $\Theta_{n,k+1}$, then the whole disjunction is, so we can assume that ϕ has the form $\exists y\phi'$ or $\forall y\phi'$ for some formula ϕ' with quantifier-rank k and $n+1$ free variables. Then by induction ϕ' can be expressed as a disjunction from $\Theta_{n+1,k}$, say by $\bigvee_{i\in X}\theta_i$, for some finite set $X \subseteq m$, with $|\Theta_{n+1,k}| = m$. Then $\exists y\phi'$ is equivalent to $\bigvee_{i\in X}\exists y\theta_i$, which is equivalent to $\bigvee_{i\in X}\left(\bigvee_{Y\subseteq m, i\in Y}\bigwedge_{j\in Y}\exists y\theta_j \wedge \forall y\bigwedge_{j\notin Y}\neg\theta_j\right)$, and since for each fixed i and Y the expression is an element of $\Theta_{n,k+1}$, the disjunction is a disjunction of elements of $\Theta_{n,k+1}$. $\forall y\phi'$ is $\forall y\bigvee_{i\in X}\theta_i$, and since $\forall y\bigvee_{i\in m}\theta_i$ is logically true, and $\forall y(\theta_i \to \neg\theta_j)$ is true for all $i,j < m$, we have $\forall y\bigvee_{i\in X}\theta_j$ is equivalent to $\forall y\bigwedge_{i\notin X}\neg\theta_i$, and also to $\bigvee_{i\in X}\exists y\theta_i$, and so $\forall y\phi'$ is equivalent to $\bigvee_{i\in X}\exists y\theta_i \wedge \forall y\bigwedge_{i\notin X}\neg\theta_i$, which is in $\Theta_{n,k+1}$.

3.3.2. Show that in the statement of Theorem 3.3.2, the formulas in $\Theta_{n,k}$ can all be taken to be $\exists_{k+1}$ formulas. Show also that they can all be taken to be $\forall_{k+1}$ formulas.

We show this by simultaneous induction. It is clear for $k = 0$. At $k+1$, every term is of the form $\bigvee_{i\in X}\exists x_n\chi_i(x_0,\ldots,x_n) \wedge \forall x_n\bigwedge_{i\notin X}\neg\chi(x_0,\ldots,x_n)$. By induction, we can assume that each χ_i with $i \in X$ is $\exists_k$, and each with $i \notin X$ is $\forall_k$. Thus we have a disjunction of $\exists_k$ statements along with a conjunction of $\forall_k$ statements. We can pull all of the quantifiers out of the disjunction, preserving the $\exists_k$ (at the cost of increasing the total number of quantifiers), and likewise with the disjunction. Then we have $\exists\bar{x}_1\forall\bar{x}_2\ldots\theta_1 \wedge \forall\bar{y}_1\exists\bar{y}_2\ldots\theta_2$, the conjunction of $\exists_k$ and $\forall_k$ formulas. Then, pulling the $\forall$ quantifier out first gets us an $\forall_{k+1}$ formula, and pulling the $\exists$ quantifier out first yields an $\exists_{k+1}$ formula.

3.3.3. Show that every unnested first-order formula of quantifier rank k is logically equivalent to an unnested first-order formula of quantifier rank k which is in negation normal form.

Problem 2.6.2 shows that every first-order formula is equivalent to one in negation normal form. Observing that proof, it is clear that if ϕ is unnested, so is the negation normal form, and the operations performed do not raise the quantifier rank, so the negation normal form has the same quantifier rank as ϕ.

3.3.4. Find a simple set of axioms for $\mathrm{Th}(\mathbb{Z},+,<)$.

Let $\tilde{\bigvee}$ denote exclusive or. We construct a set of axioms, T. T has axioms that $\mathbb{Z}$ is an ordered group with least positive element, 1, and for every positive integer n and for all x, exactly one of $x, x+1, \ldots, x+n-1$ is divisible by n. We write this as $\forall x \tilde{\bigvee}_{i<n} \exists y\,(ny = x+i)$, for each n. Note that this allows us to define $\pmod{n}$ for any model of T. Then the proof of Lemma 3.3.7 goes through as before, yielding the same elimination set. This allows us to play a back-and-forth game between any two models of T, showing that they are back-and-forth equivalent, and thus elementarily equivalent. Therefore T is a set of axioms for $\mathrm{Th}(\mathbb{Z},+,<)$.

3.3.5*. Let L be the first-order language of linear orderings. (a) Show that if $h < 2^k$ then there is a formula $\phi(x,y)$ of L of quantifier rank $\leq k$ which expresses (in any linear ordering) '$x < y$ and there are at least h elements strictly between x and y. (b) Let A be the ordering of the integers, and write $s(a,b)$ for the number of integers strictly between a and b. Show that if $a_0 < \ldots < a_{n-1}$ and $b_0 < \ldots < b_{n-1}$ in A, then $(A, a_0, \ldots, a_{n-1}) \approx_k (A, b_0, \ldots, b_{n-1})$ iff for all $m < n-1$ and all $i < 2^k$, $s(a_m, a_{m+1}) = i \leftrightarrow s(b_m, b_{m+1}) = i$.

(a) For arbitrary i, let j be the unique number such that $i = j+1+j$, or $i = j+1+(j+1)$ (depending if i is odd or even), and let $j^* = j$ in the first case and $j+1$ in the second. Then we can build up ϕ_h as follows. ϕ_0 is $x < y$. ϕ_i is $\exists z\,(\phi_j(x,z) \wedge \phi_{j^*}(z,y))$. By induction, ϕ_i has quantifier rank greater than or equal to ϕ_j for $i > j$, so we need only verify the quantifier rank of ϕ_{2^k-1}. It is equal to 1 plus the quantifier rank of $\left(2^k-2\right)/2 = 2^{k-1}-1$, so by induction, it has rank $< k$.

(b) The statement is incorrect – it is sufficient that the condition hold for $i < 2^k - 1$. Since we can express that a gap is at least i for any $i < 2^k - 1$ with an unnested formula with quantifier rank at most $k-1$, it is clear that the condition is necessary. To show that it is sufficient, we give a strategy for $\exists$, given such a setup and show that it works by induction on k. The case $k=0$ is trivial, since $\exists$ has already won. If $\forall$ chooses an element c in an interval (a_m, a_{m+1}) with $s(a_m, a_{m+1}) < 2^k - 1$, then by assumption, $s(b_m, b_{m+1}) = s(a_m, a_{m+1})$, and there is a natural choice for $\exists$, d, which preserves 0-equivalence and also the conditions on gaps, so $(A, \bar{a}, c) \approx_{k-1} \left(B, \bar{b}, d\right)$, by induction. If $\forall$ chooses an element greater than all the a_i's, $\exists$ also has an easy choice of an element equally far from the b_i's, and likewise for less than. It remains to consider the case when $s(a_m, a_{m+1}) \neq s(b_m, b_{m+1})$ and $\forall$ chooses an element in (a_m, a_{m+1}), c. We know that either $s(a_m, c) \geq 2^{k-1}-1$ or $s(c, a_{m+1}) \geq 2^{k-1}-1$ (possibly both). If it is both, then choose an element of (b_m, b_{m+1}) which is at least $2^{k-1}-1$ from both b_m and b_{m+1} – possible since $s(b_m, b_{m+1}) \geq 2^k - 1$. If $s(a_m, c) < 2^{k-1}-1$, then choose d in (b_m, b_{m+1}) such that $s(b_m, d) = s(a_m, c)$, possible for the same reason. Note that $s(c, a_{m+1}) > 2^{k-1}-1$, and $s(d, b_{m+1}) > 2^{k-1}-$. The remaining case is similar. Now by induction, we are done, since the conditions for $k-1$ are satisfied.

3.3.6. Show that if G, G', H and H' are groups with $G \preccurlyeq G'$ and $H \preccurlyeq H'$, then $G \times H \preccurlyeq G' \times H'$.

Any formula with parameters in $G \times H$ is easily seen to be equivalent to a conjunction of a formula in G and a formula in H, so the Tarski-Vaught criterion shows that $G \times H \preccurlyeq G' \times H'$.

3.3.7. Show that there is no formula of first-order logic which expresses '$\langle a, b\rangle$ is in the transitive closure of R', even on finite structures. (For infinite structures it is easy to show there is no such formula.)

$\langle a, b\rangle$ is in the transitive closure of R iff there are $c_0, \ldots, c_m$, for some m, with $R(a, c_0) \wedge \bigwedge_{i<m} R(c_i, c_{i+1}) \wedge R(c_m, b)$. Suppose there were a first-order way to express this, say by $\varphi(x, y)$. Let the quantifier rank of φ be r. Then it is easy to see that in the model A with $R = \{(c_i, c_{i+1}) \mid i < 2^r\} \cup \{(d_i, d_{i+1}) \mid i < 2^r\}$, $(A, c_0, d_{2^r}) \approx_r (A, c_0, c_{2^r})$, but then φ must be true (or false) on both, which is impossible.

3.3.8*. Let A and B be structures of the same signature. Immerman's pebble game on A, B of length k with p pebbles is played as follows. Pebbles $\pi_0, \ldots, \pi_{p-1}, \rho_0, \ldots, \rho_{p-1}$ are given. The game is played like $\mathrm{EF}_k[A, B]$, except that at each step, player $\forall$ must place one of the pebbles on his choice (one of the π_i if he chose from A, one of the ρ_i if he chose from B), then player $\exists$ must put the corresponding ρ_i (π_i) on her choice. (At the beginning the pebbles are not on any elements; later in the game the players may have to move pebbles from one element to another.) The condition for player $\exists$ to win is that after every step, if $\bar{a} = (a_0, \ldots, a_{p-1})$ is the sequence of elements of A with pebbles $\pi_0, \ldots, \pi_{p-1}$ resting on them (where we ignore any pebbles not resting on an element), and likewise $\bar{b} = (b_0, \ldots, b_{p-1})$ the elements of B labelled by $\rho_0, \ldots, \rho_{p-1}$, then for every unnested atomic $\phi(x_0, \ldots, x_{p-1})$, $A \models \phi(\bar{a}) \Leftrightarrow B \models \phi(\bar{b})$. Show that player $\exists$ has a winning strategy for this game if and only if A and B agree on all first-order sentences which have quantifier rank $\leq k$ and use at most p distinct variables.

[This problem doesn't make sense to me. The pebble game should be weaker than $\mathrm{EF}_k[A, B]$, not stronger. I would think that "quantifier rank" should instead by "number of quantifiers," but the problem as stated that way is false.]

I assume that the first-order sentences in question are assumed to be unnested, since there is no bound to their complexity if not. I assume the language is finite, because the problem is false otherwise (models with infinitely many unary predicates, and every finite boolean combination realized). It is easy to see that $p \geq k$, so if $\exists$ can win the pebble game, clearly $\exists$ can win $\mathrm{EF}_k[A, B]$, so A and B agree on all unnested sentences of quantifier rank $\leq k$, whatever their variables. Conversely, if A and B agree on all unnested sentences with quantifier rank $\leq k$ and at most p variables, we show that $\exists$ has a winning strategy. Let $(\bar{a}, \bar{b})$ be the play so far, with the condition for $\exists$ preserved so far. Let $\bar{a}$ have length m. As well, let $A \models \psi(\bar{a}) \Leftrightarrow B \models \psi(\bar{b})$ for every formula ψ with quantifier rank at most $k - m$ and using at most p distinct variables (true at $m = 0$). Suppose $\forall$ takes a new pebble π and puts it on some $c \in A$. Then, since there are only finitely many unnested formulas using at most p distinct variables, we can conjunct all such formulas with quantifier rank at most $k - m - 1$ which $\bar{a}c$

satisfy, as $\varphi(\bar{a}, c)$. By renaming the variables, we can assume that φ still has at most p variables. Then $A \models \exists x \varphi(\bar{a}, x)$, so $B \models \exists x \varphi(\bar{a}, x)$, and so $\exists$ can choose the witness in B. This preserves the desired properties. Now suppose $\forall$ moves an already placed pebble. The same argument applies. Thus, $\exists$ can make k moves, and therefore win.

3.3.9. Let A and B be structures of the same signature. Show that A is back-and-forth equivalent to B if and only if player $\exists$ has a winning strategy for the game $\mathrm{EF}_\omega[A, B]$.

The forward direction is trivial. In the reverse, since every formula is equivalent to an unnested formula, if $\mathrm{EF}_\omega[A, B]$, then let $\bar{a}$, $\bar{b}$ be the play. If $(A, \bar{a}) \not\equiv_0 (B, \bar{b})$, then there is some formula on which they disagree. Translating that formula into unnested form, we have a contradiction.

4.1.1. Show that for every abstract group G there is a structure with domain G whose automorphism group is isomorphic to G.

Let X be a set of generators of G, and for each $x \in X$ let $f_x : G \to G$ be the function $g \to g \cdot x$. Consider the structure A with universe G and functions f_x. Any element of G induces an automorphism of A: if $g \in G$, then $f_x(ga) = gax = g(ax) = g(f_x a)$. Moreover, let φ be any automorphism of A. Suppose $\varphi(a) = b$. Since G is a group, there is a unique element g such that $ga = b$. Now consider any $c \in A$. There is some d such that $ad = c$, so writing d as a product of generators $x_1, \ldots, x_n$, $f_{x_n}(\ldots f_{x_1}(a) \ldots) = c$. Then $\varphi(c) = f_{x_n}(\ldots f_{x_1}(b) \ldots) = bd = gad = gc$, so φ and g are identical.

4.1.2. Show that if the structure B is an expansion of A, then there is a continuous embedding of $\mathrm{Aut}(B)$ into $\mathrm{Aut}(A)$. (b) Show that if B is a definitional expansion then this embedding is an isomorphism.

Clearly any automorphism of B is an automorphism of A, so there is a natural embedding. This embedding is trivially continuous, since the basic open sets are defined on tuples, and the universes of B and A are the same.

(b) Let φ be any automorphism of A. Then φ is an automorphism of B, since if φ preserves formulas, and every new symbol in B is defined by some formula. Thus, $\mathrm{Aut}(B) = \mathrm{Aut}(A)$.

4.1.3. Show that if G is a group of permutations of a set Ω, $\bar{a}$ is a tuple of elements of Ω and h is a permutation of Ω, then $G_{(h\bar{a})} = h\left(G_{(\bar{a})}\right)h^{-1}$.

Let k be an element of $G_{(h\bar{a})}$. Then clearly $h^{-1}kh$ fixes $\bar{a}$, so $h^{-1}kh \in G_{(\bar{a})}$. Thus, $h\left(h^{-1}kh\right)h^{-1} \in h\left(G_{(\bar{a})}\right)h^{-1}$, so $k \in h\left(G_{(\bar{a})}\right)h^{-1}$. Thus, $G_{(h\bar{a})} \subseteq h\left(G_{(\bar{a})}\right)h^{-1}$. Likewise, if $k \in h\left(G_{(\bar{a})}\right)h^{-1}$, say $k = hgh^{-1}$, with $g \in G_{(\bar{a})}$, then $k(h\bar{a}) = hg\bar{a} = h\bar{a}$, so $k \in G_{(h\bar{a})}$.

4.1.4. Show that if G is an oligomorphic group of permutations of a set Ω, and X is a finite subset of Ω, then $G_{(X)}$ is also oligomorphic.

Fix n. We must show that there are only finitely many orbits of n-tuples. Suppose not. Then there must be infinitely many distinct orbits. Let $\{\bar{a}_i \mid i < \omega\}$ be representatives of these orbits. Let $\bar{b}$ be a listing of X. Consider $\{\bar{b}\bar{a}_i \mid i < \omega\}$. Since G is oligomorphic, there must be infinitely many of these elements in one orbit of G. Thus, we can assume that they are all in the same orbit. Then for each i, there is a g_i, $g_i\bar{b} = \bar{b}$ and $g_i\bar{a}_0 = g\bar{a}_i$. But then $g_i \in G_{(X)}$, and $g_i\bar{a}_0 = \bar{a}_i$, so in fact $\{\bar{a}_i \mid i < \omega\}$ has only finitely many orbits in it, contradiction.

4.1.5. Suppose G is a subgroup of $\mathrm{Sym}(\Omega)$. (a) Show that the topology on G is Hausdorff. (b) Show that the basic open sets are exactly the right cosets of basic open subgroups; show that they are also exactly the left cosets of basic open subgroups.

(a) Let g and h be distinct elements of G. Then for some $a \in \Omega$, $g(a) \neq h(a)$. Then $S(a, ga)$ has empty intersection with $S(a, ha)$, and the first set contains g and the second h.

(b) Let $S = S\left(\bar{a}, \bar{b}\right) \cap G$ be a basic open set. Let $H = G_{(\bar{a})}$. If $g, k \in S$, then $g^{-1}k \in H$, so $k \in Hg$, so S is contained in a right coset of H. As well, if $k \in Hg$, then $k = hg$, for some h, so $k\bar{a} = \bar{b}$, so $k \in S\left(\bar{a}, \bar{b}\right)$, so the two are equal. Conversely, if H is a basic open subgroup of G, then $H = G_{(\bar{a})}$, for some $\bar{a}$. Let Hg be any right coset of H, and let $g\bar{a} = \bar{b}$. Then $Hg = S$, by the same argument. The left coset argument just considers $H = G_{\bar{b}}$ and $gk^{-1} \in H$.

4.1.6. Show that every open subgroup of $\mathrm{Aut}(A)$ is closed. Give an example to show that the converse fails.

Let G be an open subgroup of $\mathrm{Aut}(A)$. Let h be an element such that for any $\bar{a}$ in A, there is a $g \in G$ with $h\bar{a} = g\bar{a}$. Since G is open, it contains the pointwise stabilizer of some $\bar{a}$. Let $g \in G$ be such that $g\bar{a} = h\bar{a}$. Then consider $k = g^{-1}h$. $k \in S(\bar{a}, \bar{a})$, so $k \in G$. Then $gk = h$, so $h \in G$. For the converse, consider the trivial subgroup consisting of the identity permutation. It is certainly not open, but it is closed.

4.1.7. Show that a subgroup of $\mathrm{Aut}(A)$ is open if and only if it has non-empty interior.

A subgroup G has non-empty interior if and only if it contains an open set. Then it contains a basic open set, say $S\left(\bar{a}, \bar{b}\right)$. But then it is easy to see that it contains $S(\bar{a}, \bar{a})$. Then G is easily a union of cosets of $S(\bar{a}, \bar{a})$, and thus open. The converse is trivial.

4.1.8. Show that if A is an infinite structure then $\mathrm{Aut}(A)$ has a dense subgroup of cardinality at most $\mathrm{card}(A)$.

We show that $\mathrm{Sym}(\Omega)$ has a dense subgroup of cardinality $\leq |A|$. Then, since $\mathrm{Aut}(A)$ is closed, this subgroup will be dense in $\mathrm{Aut}(A)$. Note that there are $|A|^{<\omega} = |A|$ basic open sets. Choose a representative from each of them. We have a set of size $|A|$. Now, we are in the language of groups, so there is a subgroup containing these elements and their inverses and closed under multiplication of size $|A|$. This is the desired group.

4.1.9. Suppose K, H, and G are subgroups of $\mathrm{Sym}(\Omega)$ with K a dense subgroup of H and H a dense subgroup of G. Show that K is a dense subgroup of G.

Let $S\left(\bar{a},\bar{b}\right)$ be any basic open set with $S\left(\bar{a},\bar{b}\right)\cap G\neq\emptyset$. Then since H is dense in G, there is some $h\in S\left(\bar{a},\bar{b}\right)\cap H$. Then since K is dense in H, there is some $k\in S\left(\bar{a},\bar{b}\right)\cap K$. Then $k\in S\left(\bar{a},\bar{b}\right)\cap G$, so K is dense in G.

4.1.10. Show that if A is a countable structure which is $L_{\omega_1\omega}$-equivalent to some uncountable structure, then A has 2^ω automorphisms.

Let B be A together with constants naming every element of A. We wish to know how many different expansions of A are isomorphic to B. Suppose there are less than 2^ω. Then for some tuple $\bar{a}$ in A, for every atomic formula $\phi(\bar{x})$ with parameters from A, there is a formula $\psi(\bar{x},\bar{y})\in L_{\omega_1\omega}$ with $B\models\forall\bar{x}\left(\phi(\bar{x})\leftrightarrow\psi(\bar{x},\bar{a})\right)$, so $A\models\forall\bar{x}\left(\phi(\bar{x})\leftrightarrow\psi(\bar{x},\bar{a})\right)$. Let ψ_a be the formula when ϕ is $x=a$, for each $a\in A$. Then $A\models\forall x\bigvee_{a\in A}\psi_a(x)$, and $A\models\exists!x\psi_a(x)$, for each $a\in A$. Both these statements are $L_{\omega_1\omega}$. But any model satisfying them is countable. Thus, A is not $L_{\omega_1\omega}$-equivalent to any uncountable model.

4.1.11. Show that if A is a countable structure and every orbit of $\mathrm{Aut}(A)$ on elements of A is finite, then $|\mathrm{Aut}(A)|$ is either finite or 2^ω.

If the number of orbits is finite, then clearly $|\mathrm{Aut}(A)|$ is finite. If the number of orbits is infinite, then clearly no finite tuple can make A rigid, so there are 2^ω automorphisms.

4.1.12. Let G be a closed subgroup of $\mathrm{Sym}(\omega)$ and H any subgroup of G. (a) Show that the closure of H in G is a subgroup of G. (b) Show that if $(G:H)<2^\omega$ then there is some tuple $\bar{a}$ such that $H_{(\bar{a})}$ is a dense subgroup of $G_{(\bar{a})}$.

(a) We must show that if $g\in\mathrm{cl}(H)$, then $g^{-1}\in\mathrm{cl}(H)$, and if $g,k\in\mathrm{cl}(H)$, then $gk\in\mathrm{cl}(H)$. For the first claim, let $\bar{a}$ be any tuple. Let $\bar{b}=g^{-1}\bar{a}$. Since $g\in\mathrm{cl}(H)$, there is some $h\in H$ with $h\bar{b}=\bar{a}$. Then $h^{-1}\bar{a}=\bar{b}$. Thus, for any tuple, we can find an element of H agreeing with g^{-1} on that tuple, so g^{-1} is in the closure of H. If $g,k\in\mathrm{cl}(H)$, consider gk and let $\bar{a}$ be any tuple. There is $h_1\in H$ with $h_1\bar{a}=k\bar{a}$, and $h_2\in H$ with $h_2(h_1\bar{a})=g(h_1\bar{a})$. Then $gh\bar{a}=h_2h_1\bar{a}$, so $gh\in\mathrm{cl}(H)$ as well. Thus, $\mathrm{cl}(H)$ is a subgroup of G.

(b) We know that if $(G:\mathrm{cl}(H))<2^\omega$, then there is some tuple $\bar{a}$ such that $G_{(\bar{a})}\subseteq\mathrm{cl}(H)$. Since $\mathrm{cl}(H)\supseteq H$, if $(G:H)<2^\omega$, then certainly $(G:\mathrm{cl}(H))<2^w$, so there is such a tuple. Then, since H is dense in $\mathrm{cl}(H)$ and $G_{(\bar{a})}$ is a subset of $\mathrm{cl}(H)$, H is dense in $G_{(\bar{a})}$.

4.2.1. Show that if A is a relativised reduct of B and B is a relativised reduct of C, then A is a relativised reduct of C.

Let L^+ be the language of C, L' the language of B, and L the language of A. Then for some $\theta(x)\in L^+$, B is the substructure of $C|L'$ with domain $\theta(C)$, and for some $\psi(x)\in L'$, A is the substructure of $B|L$ with domain $\psi(B)$. Then I claim A is the substructure of $C|L$ with domain $\theta\wedge\psi(C)$. Clearly the universe is correct. Given $\bar{a}\in A$, and R a relation in L, $A\models R(\bar{a})\leftrightarrow B\models R(\bar{a})$, since A is a relativized reduct of B, and $B \models R(\bar{a}) \leftrightarrow C \models R(\bar{a})$, since B is a relativized reduct of C. Thus, $A \models R(\bar{a}) \leftrightarrow C \models R(\bar{a})$. The same works for constants and functions, so the claim is proved.

4.2.2. Describe the admissibility formulas for relativisation to P.

For each constant symbol c in L, we have $P(c)$. For each function symbol f in L, we have $\bigwedge_{y \in \bar{x}} P(y) \rightarrow P(f(\bar{x}))$. There are no requirements for relation symbols.

4.2.3. Prove Corollary 4.2.2. [Let L and L^+ be signatures with $L \subseteq L^+$ and P a 1-ary relation symbol in $L^+ \setminus L$. If A and B are L^+-structures such that $A \preccurlyeq B$ and A_P is well-defined, then B_P is well-defined and $A_P \preccurlyeq B_P$.]

From the previous problem, we know that B_P is well-defined. Let $A_P \models \phi(\bar{a})$, with $\phi \in L$. Then $A \models \phi^P(\bar{a})$, by Theorem 4.2.1. Thus, $B \models \phi^P(\bar{a})$, so $B_P \models \phi(\bar{a})$, again by the theorem. The reverse works as well, when the parameters come from A_P. Thus, $A_P \preccurlyeq B_P$.

4.2.4. In Example 1, write out a formula $\psi(x, y, z)$ which expresses that the matrix z is the product of the matrices x and y.

$\psi(x, y, z)$ is

$$group(x) \wedge group(y) \wedge group(z) \wedge \bigwedge_{1 \le i,j \le n} coeff_{ij}(z) = \sum_{k=1}^{n} coeff_{ik}(x)\, coeff_{kj}(y)$$

4.2.5. Show that the structure $(\omega, +)$ is a relativised reduct of the ring of integers.

Let $\theta(x)$ be $\exists y, z, u, v\left(y^2 + z^2 + u^2 + v^2 = x\right)$. Since every non-negative integer is expressible as the sum of four squares, and clearly only non-negative integers are, θ defines ω. Then, along with the reduct of the language of rings to the language with just $+$, $(\omega, +)$ is a relativized reduct of the ring of integers.

4.2.6. Show that if B is a relativized reduct of A, then there is an induced continuous homomorphism $h : \mathrm{Aut}(A) \rightarrow \mathrm{Aut}(B)$.

Let φ be any automorphism of A. Then φ restricted to B is clearly an automorphism of B. This restriction map is easily a homomorphism and continuous.

4.2.7. Show that the downward Löwenheim-Skolem theorem holds for PC_Δ classes in the following sense: if $L \subseteq L^+$, U is a theory in L^+ and $\mathbf{K}$ is the class of all L-reducts of models of U, then for every structure A in $\mathbf{K}$ and every set X of elements of A, there is an elementary substructure of A of cardinality $\le |X| + |L^+|$ which contains all the elements of X.

I assume that this elementary substructure must also be in $\mathbf{K}$, since otherwise the problem is trivial. Let B be the model whose L-reduct A is. By the downward Löwenheim-Skolem theorem, given X in A, we can find an elementary substructure of B, B', containing X and with size $\le |X| + |L^+|$. Then by Corollary 4.2.2, the L-reduct of B' exists (so is in $\mathbf{K}$), and is an elementary substructure of A, and so is the desired structure, since it certainly contains X.

4.2.8. In Example 5, show that each ordering ζ^α is isomorphic to the reverse ordering $(\zeta^\alpha)^*$.

ζ^α has as elements sequences of integers of length α with only finitely many non-zero elements. Given a sequence $m = (m_i \mid i < \alpha)$, let $-m = (-m_i \mid i < \alpha)$. Consider the map $f : \zeta^\alpha \to (\zeta^\alpha)^*$ defined by $f(m) = -m$. I claim f is an isomorphism. First, it is clear that f is bijective. Now, suppose $m < n$ in ζ^α. Then at the last coordinate at which m and n differ, $m_i < n_i$. Thus, $-m_i > -n_i$. Therefore, in the ζ^α-ordering, $f(m) > f(n)$, and thus in the reversed ordering, $f(m) < f(n)$. Since f preserves the only relation and is bijective, it is an isomorphism.

4.2.9*. Show that the class of multiplicative groups of real-closed fields is first-order axiomatisable.

I claim this is false. Since the theory of real-closed fields is complete, we can just look at $\mathbb{R}$ for characteristics that the multiplicative group must satisfy. The elements of $\mathbb{R}$ with $\exists y\left(y^2 = x\right)$ form an abelian, divisible, torsion-free group. Moreover, for every x there is a unique y such that $y^2 = x^2$, but $x \neq y$, so the full group structure is definable from the positive part. But any two abelian, divisible, torsion-free groups are elementarily equivalent, by an argument similar to the one in Exercise 7.4.11 (elimination of quantifiers), so if the class is first-order axiomatizable, then every abelian, torsion-free, divisible group is the multiplicative group of a real-closed field. However, consider $\mathbb{Q}$ as an additive group. Then $\mathbb{Q}$ fulfills the above conditions. Thus, $\mathbb{Q}$ must be isomorphic to the multiplicative group of some real-closed field, R. Clearly $1 \in R$ maps to 0. Let $2 \in R$ map to $a \in \mathbb{Q}$. Then pa/q maps to the unique positive q-th root of 2^p, for any $p, q \in \mathbb{Z}$. But we have just mapped all of $\mathbb{Q}$ onto a proper subset of R, since 3 was not hit. Thus, $\mathbb{Q}$ cannot be isomorphic to R.

4.2.10. Show that in Theorem 4.2.3(b) the condition that all structures in $\mathbf{K}$ are infinite can't be dropped.

Let L be the language of rings along with a unary relation, P. Let $\theta(x)$ be a formula which is not r.e. in the natural numbers and has infinitely many realizations. Let T be PA without induction along with the statements $\exists! x \forall y < x\,(Py \wedge \neg Px \wedge \theta(x))$ and $\forall x > 0\,(Px \to P(x-1))$. Then T can be written as a single sentence. Let $\mathbf{K}$ be the class of relativized reducts of P on models of T to the empty language. Then $A \in \mathbf{K}$ can have size any natural number such that $\theta(x)$. Let L' be any language, and ψ any sentence in L'. Whether $\psi \vdash \exists_{=n} x\,(x = x)$ depends only on the symbols in ψ, so we can assume L' is finite. Then we have a recursive language and a recursive theory, so the consequences are r.e. Thus, the set of sizes of models of ψ is r.e. But the set of sizes of models in $\mathbf{K}$ is not r.e., and so $\mathbf{K}$ cannot be a reduct of models of ψ, and so is not PC.

4.3.1. Show that if Γ is an n-dimensional interpretation of L in K, then for every K-structure A for which ΓA is defined, $|\Gamma A| \leq |A|^n$.

Since ΓA is n-dimensional, the domain formula, $\partial_\Gamma(\bar{x})$ takes n arguments. Thus, $|\partial_\Gamma(A^n)| \leq |A^n|$. But ΓA has universe equivalence classes of ∂_Γ, so $|\Gamma A| \leq |\partial_\Gamma(A^n)| \leq |A|^n$.

4.3.2. Let A, B, and C be structures. Show that if B is interpretable in A and C is interpretable in B then C is interpretable in A.

Let the languages of A, B, C be L^+, L', and L, respectively. Let C be interpretable in B with the formula $\partial^C(\bar{x})$, a translation of $\phi(y_1, \ldots, y_k)$ in L to $\phi_C(\bar{x}_1, \ldots, \bar{x}_k)$ in L' for every unnested atomic formula of L, and a surjective map f_C from n-tuples of B to elements of C. Let similar objects be defined for the interpretation of B in A as an m-dimensional interpretation. Now, let $\partial_\Gamma(\bar{x}_0, \ldots, \bar{x}_{n-1})$, which takes n m-tuples as arguments, be $\bigwedge_{i<n} \partial^B(\bar{x}_i) \wedge \partial_B^C(\bar{x}_0, \ldots, \bar{x}_{n-1})$, let $\phi(y_1, \ldots, y_k)$ in L an atomic formula be mapped to $\phi_{CB}(\bar{x}_{1,0}, \ldots, \bar{x}_{1,n-1}, \ldots, \bar{x}_{k,0}, \ldots, \bar{x}_{k,n-1})$, with each $\bar{x}_{i,j}$ an m-tuple, and let $f_\Gamma : A^{mn} \to C$ be $f_\Gamma(\bar{a}_0, \ldots, \bar{a}_{n-1}) = f_C(f_B\bar{a}_0, \ldots, f_B\bar{a}_{n-1})$. Note that since f_B is surjective and f_C is surjective, f_Γ is surjective. We show that Γ is actually an interpretation of C in A by showing that for any unnested atomic formula ϕ of L and all $\bar{a}_i \in \partial_\Gamma(A^{mn})$, $C \models \phi(f_\Gamma\bar{a}_1, \ldots, f_\Gamma\bar{a}_k) \Leftrightarrow A \models \phi_\Gamma(\bar{a}_1, \ldots, \bar{a}_k)$. Write $f_\Gamma\bar{a}_i$ as $f_C(f_B\bar{a}_{i,0}, \ldots, f_B\bar{a}_{i,n-1})$, with each $\bar{a}_{i,j}$ an m-tuple, let $b_{i,j} = f_B\bar{a}_{i,j}$, and let $\bar{b}_i = (b_{i,0}, \ldots, b_{i,n-1})$. We know that $C \models \phi\left(f_C\bar{b}_1, \ldots, f_C\bar{b}_k\right) \Leftrightarrow B \models \phi_C\left(\bar{b}_1, \ldots, \bar{b}_k\right)$. By Theorem 4.3.1, the interpretations extend to every formula, so $B \models \phi_C\left(\bar{b}_1, \ldots, \bar{b}_k\right) \Leftrightarrow C \models \phi_{CB}(\bar{a}_1, \ldots, \bar{a}_k)$, but $\phi_{CB} = \phi_\Gamma$, so the claim is proved.

4.3.3. Write down an interpretation Γ such that for every abelian group A, ΓA is the group $A/5A$. By applying Γ to the inclusion $\mathbb{Z} \to \mathbb{Q}$, show that Theorem 4.3.3(b) fails if we replace $\exists_1^+$ by $\exists_1$ and 'homomorphisms' by 'embeddings'.

∂_Γ is just $x = x$. The map $f_\Gamma : A \to \Gamma A$ just maps a to its coset of $5A$. $=_\Gamma(x, y)$ is $\exists z\,(x = y + 5z)$. It is easy to see that $f_\Gamma(xy) = f_\Gamma x f_\Gamma y$, since the cosets form a group. Since there are no relations, this defines the interpretation. We show that this really is an interpretation. We have an unnested atomic formula ϕ, which is then $x = y$. Let a_1, a_2 be any elements of A. Then $\Gamma A \models f_\Gamma a_1 = f_\Gamma a_2$ if and only if a_1 and a_2 lie in the same coset of $5A$ – in other words, if there is some $b \in 5A$ such that $a_1 = a_2 + b$. But then, letting $5c = b$, we have that $\Gamma A \models f_\Gamma a_1 = f_\Gamma a_2 \leftrightarrow A \models \exists z\,(a_1 = a_2 + 5z)$, so we have an interpretation.

Clearly every atomic formula and ∂_Γ is $\exists_1$, and $\mathbb{Z}$ embeds into $\mathbb{Q}$. However, $\mathbb{Z}/5\mathbb{Z}$ is not an elementary substructure of $\mathbb{Q}/5\mathbb{Q}$ – since $\mathbb{Q}$ is divisible, $\mathbb{Q}/5\mathbb{Q} = \{0\}$, but $\mathbb{Z}/5\mathbb{Z}$ is certainly not trivial.

4.3.4. Let A be an L-structure with at least two elements. Show that the disjoint sum $A + A$ (see Exercise 3.2.8) is interpretable in A.

Let a and b be distinct elements of A. Let $\partial_\Gamma(x, y)$ be $x = x \wedge (y = a \vee y = b)$. Let $f_\Gamma : \partial_\Gamma\left(A^2\right) \to A \oplus \{a, b\}$ be $(x, y) \to (x, y)$. Let $\varphi(\bar{x})$ be any unnested atomic formula of L, the language of A. Let $\tilde{\varphi}(\bar{x})$ be the translation of that formula into the language of $A + A$, $\tilde{L}$ (every constant and function symbol becomes a relation). Note that every atomic formula of $\tilde{L}$ is either of this form or is Px or Qx, where P and Q are the unary predicates picking out each copy of A. If $\varphi(x)$ is Px, define $\varphi_\Gamma(y,z)$ as $z = a$, and similarly for Q. For any other $\tilde{\varphi}(x_1,\ldots,x_k)$, $\tilde{\varphi}_\Gamma(y_1,z_1,\ldots,y_k,z_k)$ is $\bigwedge_{i\leq k} z_i = a \vee \bigwedge_{i\leq k} z_i = b \wedge \varphi(y_1,\ldots,y_k)$. It is easy to see that this is an interpretation.

4.3.5. Let G be a group and A a normal abelian subgroup of G such that G/A is finite. Show that G is interpretable in A with parameters.

Let $h_1,\ldots,h_n$ be representatives of the finitely many cosets of G/A. Then any element of G can be written as $h_i a$ for some $i \leq n$, $a \in A$, and also as $h_1 a_1 + \ldots h_n a_n$ with all but one a_i 0. Then $\partial_\Gamma(x_1,\ldots,x_n)$ is $\bigvee_{i\leq n}\bigwedge_{j\leq n, j\neq i} x_j = 0$, the map $\partial_\Gamma(A^n) \to \sum_{i\leq n} h_i a_i$ is the desired f_Γ, and $=_\Gamma(y_1,\ldots,y_n,z_1,\ldots,z_n)$ is true iff $\bigwedge_{i\leq n} y_i = z_i$. Since as before the map f_Γ respects multiplication, this is clearly an interpretation of G.

4.3.6. Show how a polynomial interpretation Δ of L in K induces an interpretation Γ of L in K, in which for every equation ϕ, ϕ_Γ is also an equation. Show that for every K-structure A, ΓA exists and is a reduct of a definitional extension of A. (We write ΔA for ΓA.)

Our interpretation Γ has $\partial_\Gamma(x)$ just $x = x$, for each formula $x = c$ in L, the formula $x = c_\Delta$ in K, and for every formula $F(x_0,\ldots,x_{m-1}) = y$ in L, the formula $F_\Delta(x_0,\ldots,x_{m-1}) = y$ in K. Then Γ is clearly an interpretation, and every equation is still an equation. It is easy to see that the admissibility criteria are trivially satisfied (since c_Δ is a closed term and functions have unique valuations), so ΓA is always an L-structure when A is a K-structure. If we add all the symbols in L to K to get K^+, then A becomes a K^+-structure, with all the symbols in L satisfying Δ (so $F(x_0,\ldots,x_{m-1}) = F_\Delta(x_0,\ldots,x_{m-1})$). Thus, A as a K^+-structure is a definitional expansion of A. Then the L-reduct is just ΓA.

4.3.7. Write down a polynomial interpretation Δ such that for every ring A, ΔA is the Lie ring of A.

Define $[a,b] = ab - ba$, addition to be the same as on A, and 0 and 1 the same.

4.3.8. Let A be a field and n a positive integer. (a) Let R be the set of all n-tuples $\bar{a} = (a_0,\ldots,a_{n-1})$ of elements of A such that the polynomial $X^n + a_{n-1}X^{n-1} + \ldots + a_1 X + a_0$ is irreducible over A, and if α is a root of this polynomial then the field $A[\alpha]$ is Galois over A. Show that R is $\emptyset$-definable over A. (b) Let G be a finite group. Let R_G be the set of all n-tuples as above, such that the Galois group $A[\alpha]/A$ is isomorphic to G. Show that R_G is also $\emptyset$-definable over A.

(a) Any element of $A[\alpha]$ can be written as $\sum_{i<n} b_i\alpha^i$, for some b_is in A. Addition and multiplication are easy to define ($\alpha^n = -\sum_{i<n} a_i\alpha^i$), so we can interpret $A[\alpha]$ in A. Now, for $A[\alpha]$ to be Galois, it must be normal and separable. Thus, for every element β of $A[\alpha]$, the minimal polynomial of β with degree $m \leq n$ has exactly m roots in $A[\alpha]$. This condition, for any β, is expressible: let $c_0,\ldots,c_n$ be any $n+1$-tuple. Then $\sum_{i\leq n} c_i X^i$ is irreducible iff there do not exist $d_0,\ldots,d_n$ and $e_0,\ldots,e_n$, with not

all $d_i = 0$ $(0 < i \leq n)$, not all $e_i = 0$ $(0 < i \leq n)$, such that for all $i \leq n$, $c_i = \sum_{j+k=i} d_j e_k$. Let $\varphi(\bar{c})$ be the first-order formula expressing this. $\sum_{i \leq n} c_i \beta^i = 0$ is a condition that can be determined from our interpretation of $A[\alpha]$ in A, so let $\psi(\bar{c}, \bar{x})$ be a first-order formula expressing this, where $\bar{x}$ is an n-tuple. The disjunction of the statements $c_m \neq 0 \wedge \bigwedge_{m<i\leq n} c_i = 0 \wedge \exists_{=m} x \sum_{i \leq m} c_i x^i = 0$ says that $\bar{c}$ splits in $A[\alpha]$. Let $\theta(\bar{c})$ say this. (ψ, and θ have parameters $\bar{a}$.) Thus, $\forall \bar{x} \exists \bar{y} (\varphi(\bar{y}) \wedge \psi(\bar{y}, \bar{x}) \wedge \theta(\bar{y}))$ says that $A[\alpha]$ is Galois. Thus, this formula along with $\varphi(\bar{a})$ defines R.

(b) If the Galois group $A[\alpha]/A$ is isomorphic to G, then we can consider G as a subgroup of $\mathrm{Sym}(n)$, permuting the roots of α's minimal polynomial. Note that any automorphism is determined by where α goes, since every element is expressible using just α and elements of A, which are fixed. For each automorphism, sending α to $\sum_{i<n} b_i \alpha^i$, we form a function $f_{\bar{b}}$ taking an n-tuple interpreted as an element of $A[\alpha]$ to its image under this automorphism. Note that, given $\bar{b}$, $f_{\bar{b}}$ is definable. We can then first-order say that $f_{\bar{b}}$ is surjective, and if $\bar{c}$, $\bar{d}$ are n-tuples (interpreted as elements of $A[\alpha]$), then $f_{\bar{b}}\bar{c} + f_{\bar{b}}\bar{d} = f_{\bar{b}}\left(\bar{c} + \bar{d}\right)$, where $+$ is the addition function for $A[\alpha]$. Similarly for multiplication. Thus, we can first-order express that $\bar{b}$ defines an automorphism. Let $m = |G|$. Let $\mathrm{diag}(G|\{f_1, \ldots, f_m\})$ be the diagram of G with the constants for the elements being $f_1, \ldots, f_m$. Then, if $m = |G|$, we can say $\exists \bar{b}_1 \ldots \bar{b}_m \left(\mathrm{diag}\left(G|\{f_{\bar{b}_1}, \ldots, f_{\bar{b}_m}\}\right) \wedge \bigwedge_{i \leq m} \bar{b}_i \text{ is an automorphism}\right)$, and "there are exactly m $\bar{b}$ such that $f_{\bar{b}}$ is an automorphism." Thus we can first-order express that the Galois group of $A[\alpha]$ over A is G, and so define R_G.

4.4.1. Show that if E is an equivalence relation on tuples in a set I_θ in A^{eq} and E is definable in A^{eq} without parameters, then E is equivalent in a natural way to an equivalence relation on tuples of elements of A which is definable in A without parameters. (So there is nothing to be gained by passing to $(A^{eq})^{eq}$.)

Let $\psi(x_1, \ldots, x_k, y_1, \ldots, y_k)$ define E. Let θ be an equivalence relation on n-tuples. We can make ψ into a formula on A by considering $\psi'(z_{1,1}, \ldots, z_{1,n}, \ldots, z_{k,n}, w_{1,1}, \ldots, w_{k,n})$ (with $\bar{z}_i = (z_{i,1}, \ldots, z_{i,n})$, and the same for w) to be $\bigwedge_{i \leq n} (\partial_\theta(\bar{z}_i) \wedge \partial_\theta(\bar{w}_i)) \wedge \psi(f_\theta \bar{z}_1, \ldots, f_\theta \bar{z}_n, f_\theta \bar{w}_1, \ldots, f_\theta \bar{w}_n)$. Then we can translate ψ' into a formula of L, since its domain is A. Now ψ' defines an equivalence relation on tuples in A (two tuples are equivalent if their θ-classes are ψ-equivalent) which is the natural equivalent of E.

4.4.2. Let A be an L-structure and B a finite slice of A^{eq}. (a) Show that the restriction map $g \to g|\,\mathrm{dom}(A)$ defines an isomorphism from $\mathrm{Aut}(B)$ to $\mathrm{Aut}(A)$. (b) Show that if $\mathrm{Aut}(A)$ is oligomorphic, then so is $\mathrm{Aut}(B)$.

Any automorphism of B, since $A \subseteq B$, with all its structure, must also be an automorphism of A. Conversely, if φ is an automorphism of A, we show that we can extend it to a unique automorphism of B. Given any I_θ in B, noting that since f_θ is a homomorphism from A^n to I_θ, we can extend φ to I_θ, so if $\varphi(\bar{a}) = \bar{b}$, then $\varphi(f_\theta(\bar{a})) = f_\theta\left(\bar{b}\right)$. Note that this extension is unique, given φ. Doing this for all I_θ gives us the unique automorphism of B extending A. Thus, g is an isomorphism.

(b) As we see above, if $\bar{a}$ is mapped to $\bar{b}$ under some automorphism of A, then $f_\theta(\bar{a})$ is mapped to $f_\theta(\bar{b})$ under some automorphism of B. Thus, if there are finitely many orbits of n-tuples in A then there are finitely many orbits of singletons in I_θ. Assume $\mathrm{Aut}(B)$ is not oligomorphic, so let $\{\bar{b}_i \mid i < \omega\}$ be an infinite set of n-tuples all in different orbits. Since there are only finitely many I_θ's in B, we can find an infinite subset of ω, S, such that if $i \in S$, then $\bar{b}_i(k) \in I_\theta \leftrightarrow \bar{b}_j(k) \in I_\theta$, for every I_θ in B, every $i, j \in S$, and $k < n$ (where $\bar{b}_i(k)$ is the k-th coordinate of $\bar{b}_i$). Now form $\{\bar{a}_i \mid i \in S\}$ by, for each coordinate of $\bar{b}_i$, choosing a tuple in A such that f_θ applied to the tuple yields that coordinate of $\bar{b}_i$. Then, since A is oligomorphic, there are only finitely many orbits that the $\bar{a}_i$'s go into, but if $\bar{a}_i$ and $\bar{a}_j$ are in the same orbit, then certainly $\bar{b}_i$ and $\bar{b}_j$ are in the same orbit, since each f_θ is a homomorphism.

4.4.3. Let L be a first-order language and T a complete theory in L. Show that, if T has the finite cover property (in Shelah's [Keisler's] sense), then there is a formula $\phi(x, \bar{y})$ of L such that, for arbitrarily large finite n, T implies that there are $\bar{a}_0, \ldots, \bar{a}_{n-1}$ for which $\neg\exists x \bigwedge_{i\in n} \phi(x, \bar{a}_i)$ holds, but $\exists x \bigwedge_{i \in W} \phi(x, \bar{a}_i)$ holds for each proper subset W of n.

Since T has the finite cover property, there is some formula $\theta(\bar{x}, \bar{w}, \bar{z})$ such that there is no formula $\psi(\bar{z})$ such that for any $\bar{a}$, $\psi(\bar{a})$ implies that $\theta(\bar{x}, \bar{w}, \bar{a})$ is an equivalence relation with infinitely many classes. Thus, there must be $\bar{a}_i$ for arbitrarily large i such that $\theta(\bar{x}, \bar{y}, \bar{a}_i)$ has exactly i classes. Let $\{\bar{b}^i_j \mid j < i\}$ be these representatives. Then, letting $\bar{y} = \bar{w}\bar{z}$, taking $\phi(\bar{x}, \bar{y})$ to be $\neg\theta(\bar{x}, \bar{w}, \bar{z})$ we have the finite cover property for tuples with ϕ and witnesses $\{(\bar{b}^i_j, \bar{a}_i) \mid j < i\}$ for each i.

To reduce to the case $\exists x$ we use an induction argument on the length of $\bar{x}$. When it is 1, it is clear. Suppose for some $n > 1$, we have the finite cover property for some φ, but for $m < n$, we do not. For each $\phi(x_0, \ldots, x_{m-1}, \bar{y})$, we can find r, the least number such that if a set $\{\phi(\bar{x}, \bar{b}_i) \mid i < k\}$ is r-consistent, then it is consistent. Such an r exists: T does not have the finite cover property for $m < n$, so there is some r such that for any $k > r$, for any k tuples, $k-1$-consistency implies k-consistency. But then r-consistency implies k-consistency.

By compactness, $\varphi(x, c_1, \ldots, c_{m-1}, \bar{y})$ (with $c_1, \ldots, c_m$ new constants) has such an r, since for any assignment of $c_1, \ldots, c_m$ there is such an r, and by compactness the r must be uniformly bounded. Fix this r.

Now, for arbitrarily large n, we have $\Gamma_n = \{\varphi(\bar{x}, \bar{a}_i) \mid i < n\}$ which is n-inconsistent, but not $n-1$-inconsistent. Let $\psi(x_1, \ldots, x_m, \bar{y}_0, \ldots, \bar{y}_{r-1})$ be $\exists x_0 \bigwedge_{i<r} \varphi(\bar{x}, \bar{y}_i)$. Let $\Gamma^*_n = \{\psi(x_1, \ldots, x_m, \bar{a}_0, \ldots, \bar{a}_{r-1}) \mid \forall j < r\, (\varphi(\bar{x}, \bar{a}_j) \in \Gamma_n)\}$. Then by induction there is some point, k, past which l-consistency implies $l+1$-consistency of Γ^*_n. Note that l-consistency of Γ^n implies l/r-consistency of Γ^*_n, so if we take $n > kr$, the $n-1$-consistency of Γ_n will imply the consistency of Γ^*_n. Let $c_1, \ldots, c_m$ witness the consistency of Γ^*_n. Now consider $\{\varphi(x_0, c_1, \ldots, c_m, \bar{a}) \mid \varphi(\bar{x}, \bar{a}) \in \Gamma_n\}$. By choice of r, since this set is r-consistent (by consistency of Γ^*_n), it is n-consistent, so we can find c_0 a witness. Then $\varphi(\bar{c}, \bar{a})$ for every $\varphi(\bar{x}, \bar{a}) \in \Gamma_n$, contradicting the n-inconsistency of Γ.

4.4.5*. Show that ZFC has elimination of imaginaries. (b) Show that first-order Peano arithmetic has uniform elimination of imaginaries.

Given an equivalence class in ZFC, $\theta(\bar{x}, \bar{a})$, we can let B be the set of all elements in this equivalence class with minimal rank. Then let $\phi(\bar{x}, Y)$ be $\exists \bar{y} \in Y(\theta(\bar{x}, \bar{y})) \wedge \forall \bar{y} \in Y \bar{z} \in Y \theta(\bar{y}, \bar{z}) \wedge \forall \bar{y} \in Y \bar{z}(\theta(\bar{z}, \bar{y}) \rightarrow rank(\bar{y}) < rank(\bar{z}))$. Clearly $\phi(\bar{x}, B)$ defines the equivalence class, and B is the only element which can be used.

(b) Let $\theta(\bar{x}, \bar{a})$ be an equivalence class. Let $\prec$ be the lexicographic ordering of tuples (it is definable). PA proves that $\exists \bar{x} \theta(\bar{x}, \bar{a}) \wedge \forall \bar{y}(\bar{y} \prec \bar{x} \rightarrow \neg\theta(\bar{y}, \bar{a}))$, where $\bar{x}$ is an n-tuple. Then let $\phi(\bar{x}, \bar{y})$ be $\theta(\bar{x}, \bar{y}) \wedge \forall \bar{z}(\bar{z} \prec \bar{y} \rightarrow \neg\theta(\bar{y}, \bar{z}))$. Then letting $\bar{b}$ be the minimal element of $\theta(\bar{x}, \bar{a})$ under this ordering, $\phi\left(\bar{x}, \bar{b}\right)$ is as desired.

4.4.6. Show that if V is a vector space of dimension at least 2 over a finite field of at least 3 elements, then V doesn't have elimination of imaginaries.

Let $\theta(x, y)$ express that y is a scalar multiple of x. It is first-order because the field is finite. Now, fix any a, and consider $\theta(x, a)$. By the conditions of the problem, this set has at least two elements, and yet does not contain all elements. Let $\phi\left(x, \bar{b}\right)$ be any formula with paramaters in V. Let $\lambda \neq 1$ be in the field, and consider the automorphism $v \rightarrow \lambda v$. The set $\theta(x, a)$ is unaltered by this automorphism, so $\phi\left(x, \bar{b}\right)$ must also be unaltered, so $\phi\left(x, \lambda\left(\bar{b}\right)\right) \leftrightarrow \phi\left(x, \bar{b}\right)$, but then $\bar{b}$ is not the only solution to make $\phi(x, \bar{y}) \leftrightarrow \theta(x, a)$.

5.1.1. Show that each of the following is equivalent to the compactness theorem for first-order logic. (a) For every theory T and sentence ϕ of a first-order language, if $T \vdash \phi$ then for some finite $U \subseteq T$, $U \vdash \phi$. (b) For every theory T and sentence ϕ of a first-order languague, if T is equivalent to the theory $\{\phi\}$ then T is equivalent to some finite subset of T. (c) For every first-order theory T, every tuple $\bar{x}$ of distinct variables and all sets $\Phi(\bar{x})$, $\Psi(\bar{x})$ of first-order formulas if $T \vdash \forall \bar{x}(\bigwedge \Phi \leftrightarrow \bigvee \Psi)$ then there are finite sets $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ such that $T \vdash \forall \bar{x}(\bigwedge \Phi' \leftrightarrow \bigvee \Psi')$.

(a) Proof from compactness: Let $T^* = T \cup \{\neg\phi\}$. T^* has no model. Thus, some finite subset of it has no model, U^*. Let $U = U^* \setminus \{\neg\phi\}$. If U has no model, then clearly $U \vdash \phi$. If U has a model, then every model of it must have ϕ true, so again $U \vdash \phi$. For the converse, let $\phi = \exists x(x \neq x)$.

(b) From compactness: Every model of T is a model of ϕ, and vice versa. By (a), some finite subset of T proves ϕ. Thus, some finite subset of T proves T. Conversely, let $\phi = \exists x(n \neq x)$.

(c) From: Suppose that for every finite $\Psi' \subseteq \Psi$, $T \not\vdash \forall \bar{x}(\bigwedge \Phi \rightarrow \bigvee \Psi')$. Denote by $\neg\Psi'$ the set $\{\neg\psi \mid \psi \in \Psi'\}$. Adding new constants, $\bar{a}$, to the language, then $T \cup \Phi(\bar{a}) \cup \neg\Psi'(\bar{a})\}$ is consistent. Thus, every finite subset of $T \cup \Phi(\bar{a}) \cup \neg\Psi(\bar{a})$ is consistent, so it has a model, in which the interpretation of $\bar{a}$ contradicts the hypothesis. Thus, our assumption was wrong, and for some finite $\Psi' \subseteq \Psi$, $T \vdash \forall \bar{x}(\bigwedge \Phi \rightarrow \bigvee \Psi')$. Repeating this argument with finite subsets of Φ and $\neg \bigvee \Psi'(\bar{a})$ shows that some finite piece of Φ, Φ', implies Ψ'. But then since $\bigvee \Psi' \rightarrow \bigvee \Psi \rightarrow \bigwedge \Phi \rightarrow \bigwedge \Phi'$, we are done.

Conversely, given any first-order theory U, let T be the empty theory, Φ be U, and Ψ be $\exists x(x \neq x)$.

5.1.2*. (a) Let L be a first-order language, T a theory in L and Φ a set of sentences of L. Suppose that for all models A, B of T, if $A \models \phi \leftrightarrow B \models \phi$ for each $\phi \in \Phi$, then $A \equiv B$. Show that every sentence ψ of L is equivalent modulo T to a boolean combination ψ^* of sentences in Φ. (b) Show moreover that if L and T are recursive then ψ^* can be effectively computed from ψ.

(a) Let Φ' be all consequences of ψ in T which are boolean combinations of elements of Φ. Note that Φ' is closed under conjunctions. Consider $T \cup \Phi'\{\neg\psi\}$. If this is inconsistent, then some finite set is, so we have $T \vdash \theta \to \psi$, for some $\theta \in \Phi'$. But since $T \vdash \psi \to \theta$, by definition of Φ', then $T \vdash \theta \leftrightarrow \psi$, and we are done. So assume it is consistent, so we have a model, A. Let Φ_A be all boolean combinations of elements of Φ which are true in A. Note that Φ_A is closed under conjunctions and disjunctions. Consider $T \cup \Phi_A \cup \{\psi\}$. Assume it has no model. Then we have $T \vdash \psi \to \neg\theta$, with $\theta \in \Phi_A$. But then $\neg\theta \in \Psi'$, so $A \models \theta$, which is impossible. Thus, $T \cup \Phi_A \cup \{\psi\}$ has a model, B. For every $\phi \in \Phi$, $A \models \phi \leftrightarrow B \models \phi$, so $A \equiv B$. But $A \models \neg\psi$ and $B \models \psi$, contradiction. Thus we cannot find A, and so we were done at the beginning.

(b) Φ must be recursive for ψ^* to be effectively computed. Assuming that, make the following lists: list the consequences of $T \cup \{\psi\}$, list the boolean combinations of Φ, and for each boolean combination θ, list the consequences of T and θ. After some number of steps, we will have enumerated ψ^* in the consequences of $T \cup \{\psi\}$ and in the boolean combinations of Φ, and will have listed ψ as a consequence of ψ^*.

5.1.2. (Craig's trick) In a recursive first-order language L let T be an r.e. theory. Show that T is equivalent to a recursive theory T^*.

Write φ^n for $\varphi \wedge \ldots \wedge \varphi$ (n copies of φ). Since T is r.e., for each $\varphi \in T$, there is a finite computation putting φ in T, which can be coded by a single natural number, $n(\varphi)$. Consider $T^* = \{\varphi^{n(\varphi)} \mid \varphi \in T\}$. Then determining if a sentence is in T^* is recursive.

5.1.3. Let L be a first-order language, δ a limit ordinal (for example ω) and $(T_i \mid i < \delta)$ an increasing chain of theories in L, such that for every $i < \delta$ there is a model of T_i which is not a model of T_{i+1}. Show that $\bigcup_{i<\delta} T_i$ is not equivalent to a sentence of L.

Denote the union by T. Suppose it were, say to φ. Then by Exercise 5.1.1, we can find some finite subset of T, U, equivalent to T. Since U is finite, all of its sentences have come by some finite stage in the union, say i. Then $U \subseteq T_i$, so T_i implies T, so every model of T_i is a model of T. But let A be a model of T_i which is not a model of T_{i+1}. Then A is not a model of T. Contradiction.

5.1.4. Show that none of the following classes is first-order definable (i.e. by a single sentence; see section 2.2). (a) The class of infinite sets. (b) The class of torsion-free abelian groups. (c) The class of algebraically closed fields.

We use the previous problem. The class of infinite sets is defined by $\bigcup_{i<\omega} T_i$, with $T_i = \{\exists_{>j} x(x = x) \mid j < i\}$. Each T_i clearly has a model which is not a model of T_{i+1}, so the previous problem applies. Torsion-free abelian groups have theories $T_i = \{\forall x(x \neq 0 \rightarrow jx \neq 0) \mid 0 < j \leq i\} \cup U$ (with U the theory of abelian groups), and the desired models are $\mathbb{Z}/(i+1)\mathbb{Z}$. For the class of algebraically closed fields, let $A_0 = \mathbb{Q}$. Given A_i, let $k(i)$ be the least natural number such that A_i does not contain a root of every polynomial of degree $k(i)$ with coefficients in A_i. Adjoin roots of all polynomials of degree $k(i)$ with coefficients in A_i to get $A_i^{(0)}$, and repeat this countably many times to get A_{i+1}, which has a root of every polynomial with degree $k(i)$. Note that such $k(i)$ always exists by Galois theory, since there is always an irreducible polynomial with degree prime to $\{1, \ldots, k(i-1)\}$. Let $T_i = \forall y_0 \ldots y_{j-1} \exists x(x^j + y_{j-1}x^{j-1} + \cdots + y_0 = 0) \mid 0 < j < k(i)\} \cup U$ (with U the theory of fields). Then the previous exercise applies.

5.1.5. Let L be a first-order language and T a theory in L. (a) Suppose T has models of arbitrarily high cardinalities; show that T has an infinite model. (b) Let $\phi(x)$ be a formula of L such that for every $n < \omega$, T has a model A with $|\phi(A)| \geq n$. Show that T has a model B for which $\phi(B)$ is infinite.

(a) Consider $T \cup \{\exists_{>n} x(x = x) \mid n < \omega\}$. By compactness, this is consistent, so T has an infinite model.

(b) Consider $T \cup \{\exists_{>n} x(\phi(x)) \mid n < \omega\}$.

5.1.6. Let L be the first-order language of fields and ϕ a sentence in L. Show that if ϕ is true in every field of characteristic 0, then there is a positive integer m such that ϕ is true in every field of characteristic $\geq m$.

The theory of fields with characteristic 0 is axiomatized by the theory of fields, U, along with $\{n \neq 0 \mid 0 < n < \omega\}$. By compactness, since these axioms are inconsistent with $\neg\phi$, some finite set is. Let N be the maximum of the n's appearing in the above formulas which are in this finite set. Then $m = N + 1$ works.

5.1.7. (a) Let L be a first-order language, T a theory in L, and λ a cardinal $\geq |L|$. Show that if T is λ-categorical then T is complete. (b) Use (a) to give quick proofs of the completeness of (i) the theory of (non-empty) dense linear orderings without endpoints and (ii) the theory of algebraically closed fields of a fixed characteristic. Find two other nice examples.

(a) Suppose T is not complete. Let T_1 and T_2 be distinct completions. Let $A_1 \models T_1$, and $A_2 \models T_2$. Since $\lambda \geq |L|$, we can expand (shrink) A_1 and A_2 to have size λ. But they cannot be isomorphic, since they are not even elementarily equivalent. Thus, T is not λ-categorical.

(b) Any countable dense linear order without endpoints is isomorphic to the rationals by an easy back-and-forth argument. Any two algebraically closed fields of a fixed characteristic and cardinality ω_1 are isomorphic, again by a back-and-forth argument, since there are ω_1 transcendentals in each.

The theory of an injective unary function with no cycles is another example, since all models with cardinality ω_1 are isomorphic. The theory of an equivalence relation with infinitely many infinite classes is ω-categorical, and thus complete.

5.2.1. Let A be an L-structure and B an extension of A. Show that (a)-(c) are equivalent. (a) B is an elementary extension of A. (b) For every tuple $\bar{a}$ of elements of A, $\mathrm{tp}_A(\bar{a}) = \mathrm{tp}_B(\bar{a})$. (c) For every set X of elements of A and every $n < \omega$, $S_n(X;A) = S_n(X;B)$.

If B is an elementary extension of A, then for every $\varphi(\bar{x})$ and every tuple $\bar{a}$ in A, $A \models \varphi(\bar{a}) \leftrightarrow B \models \varphi(\bar{a})$. Thus $\mathrm{tp}_A(\bar{a}) = \mathrm{tp}_B(\bar{a})$. If B is not an elementary extension of A, then for some $\bar{a}$ in A, and some φ, $B \models \varphi(\bar{a})$ and $A \models \varphi(\bar{a})$. Then $\mathrm{tp}_A(\bar{a}) \neq \mathrm{tp}_B(\bar{b})$. For (a) implies (c), let p be any type in $S_n(X;B)$. Since $A \preccurlyeq B$, and p is realized in an elementary extension of B, $p \in S_n(X;A)$. If $p \in S_n(X;A)$, we can use amalgamation from the next section, or consider any finite subset of p, to show that p is in $S_n(X;B)$. (c) implies (b) follows since if $\varphi(\bar{x}) \in \mathrm{tp}_A(\bar{a})$, $\neg\varphi(\bar{x}) \in \mathrm{tp}_B(\bar{a})$, consider the set of 0-types over $\bar{a}$. These are distinct, which impossible.

5.2.2. Let A be an L-structure, X a set of elements of A, $\bar{a}$ a tuple of elements of A and e an automorphism of A which fixes X pointwise. Show that $\bar{a}$ and $e\bar{a}$ have the same complete type over X with respect to A.

$A \models \varphi(\bar{a}, \bar{b})$, with $\bar{b}$ in X, if and only if $A \models \varphi(e\bar{a}, e\bar{b})$, since e is an automorphism. But $e\bar{b} = \bar{b}$. Thus $A \models \varphi(\bar{a}, \bar{b}) \leftrightarrow A \models \varphi(e\bar{a}, \bar{b})$, so the types are the same.

5.2.3. Let A be the structure $(\mathbb{Q}, <)$ where $\mathbb{Q}$ is the set of rational numbers and $<$ is the usual ordering. Describe the complete 1-types over $\mathrm{dom}(A)$.

By quantifier elimination, the type is characterized once we know its ordering with respect to every element of A. Thus, every complete 1-type corresponds to an element in the completion of $\mathbb{Q} - \mathbb{R}$.

5.2.4. Let A be an algebraically closed field and C a subfield of A; to save notation I write C also for $\mathrm{dom}(C)$. (a) Show that two elements a, b of A have the same complete type over C if and only if they have the same minimal polynomial over C. (b) Show that for all n, $|S_n(C;A)| = \omega + |C|$.

(a) The forwards direction is obvious. A polynomial in a is 0 iff the minimal polynomial of a divides it, and the same is true for b. Since a and b have the same minimal polynomial, they satisfy all the same atomic formulas (which are all polynomial equations with coefficients in C), hence by quantifier elimination, all formulas, so they have the same complete type.

(b) $|S_n(C;A)| \geq \omega + |C|$, since the algebraic closure is always infinite, and there are C types realized in C. There are $\leq |C|^{<\omega}$ polynomials over C, so $|S_n(C;A)| \leq |C|^{<\omega} = \omega + C$.

5.2.5. Let A be a vector space over a field k and X a set of elements of A. Describe $S_n(X;A)$ for each $n < \omega$, and show that $|S_n(X;A)| \leq |k| + |X| + \omega$.

By quantifier elimination, a type of a singleton, a, just expresses a as being in the span of X, or says that a is not in the span of X. For an n-tuple, the same needs to be said of a_i over $X \cup \{a_0, \ldots, a_{i-1}\}$, for $i < n$. The number of elements in the span of X is $\leq |kX|^{<\omega}$, and then there is one more type, so $|S_n(X; A)| \leq |kX|^{<\omega} + 1 = |k| + |X| + \omega$.

5.3.1. Let T be the theory of dense linear orderings without endpoints. Describe the heir-coheir amalgams of models of T.

The heir-coheir amalgams of T are models A, B, C, D as in (3.3) with the following properties. In D, if an interval (b_1, b_2) (with $b_1, b_2 \in B \cup \{-\infty, \infty\}$ contains elements of C, then it contains elements of A. We show that any heir-coheir amalgam satisfies these properties, and that these properties assure an heir-coheir amalgam.

Let A, B, C, D be models as in (3.3). Choose any ψ with $D \models \psi(\bar{b}, \bar{c})$, with $\bar{b}$ in B, $\bar{c}$ in C. By quantifier elimination, ψ is a disjunction of conjunctions of quantifier-free formulas. Assume ψ is just a conjunction. Then ψ defines a partial ordering on $\bar{b}$ and $\bar{c}$. Clearly, if A has elements $\bar{a}$ satisfying the partial ordering in place of $\bar{c}$, then $B \models \psi(\bar{b}, \bar{a})$. But that is assured precisely by the above conditions. Conversely, if such elements $\bar{a}$ always exist, then the above conditions are satisfied.

5.3.2. Let T be the theory of the linear ordering of the integers. Describe the heir-coheir amalgams of models of T.

Let A, B, C, D be an heir-coheir amalgam. Choose any ψ with $D \models \psi(\bar{b}, \bar{c})$, as above. Using additive notation, if we have $c = b + n$, for some $n \in \mathbb{Z}$, then we know that $b \in C$ and $c \in B$, so actually $b, c \in A$. Thus, we know that in D, elements of B and C are never in the same $\mathbb{Z}$-chain unless they are in A. Now, by back-and-forth arguments on discrete linear orderings without endpoints, an elimination set for this theory is the formulas $\{x - y = n \mid n \in \mathbb{Z}\} \cup \{x < y, x = y\}$. Since equations of the form $c = b + n$ mean that both of these are in A, all we have left is a linear ordering of the c_i's between b_i's which have infinite gaps between them. So the requirements are that for $b_1, b_2 \in B \cup \{-\infty, \infty\}$, if there is an element of C between them, then there is an element of A between them, and if b and c are separated by a finite distance, then they are both in A.

5.3.3. Show that the following are equivalent, given the amalgam (3.3) above. (a) The amalgam is heir-coheir. (b) For every tuple $\bar{b}$ in B, $\mathrm{tp}_D(\bar{b}/C)$ is an heir of $\mathrm{tp}_D(\bar{b}/A)$. (c) For every tuple $\bar{c}$ in C', $\mathrm{tp}_D(\bar{c}/B)$ is a coheir of $\mathrm{tp}_D(\bar{c}, A)$.

We go (a) to (b) to (c) to (a). Let $\varphi(\bar{x}, \bar{c})$ be any formula in $\mathrm{tp}_D(\bar{b}/C)$. Then $D \models \varphi(\bar{b}, \bar{c})$. Thus, for some $\bar{a}$ in A, $B \models \varphi(\bar{b}, \bar{a})$, so $D \models \varphi(\bar{b}, \bar{a})$. Now let $\varphi(\bar{b}, \bar{x})$ be any formula in $\mathrm{tp}_D(\bar{c}/B)$. Since $D \models \varphi(\bar{b}, \bar{c})$, and $\mathrm{tp}_D(\bar{b}/C)$ is an heir of $\mathrm{tp}_D(\bar{b}/A)$, there is $\bar{a}$ in A such that $\varphi(\bar{y}, \bar{a}) \in \mathrm{tp}_D(\bar{b}/A)$, but then $D \models \varphi(\bar{b}, \bar{a})$. Finally, let $D \models \varphi(\bar{b}, \bar{c})$. Since $\mathrm{tp}_D(\bar{c}/B)$ is a coheir of $\mathrm{tp}_D(\bar{c}/A)$, we can find $\bar{a}$ such that $D \models \varphi(\bar{b}, \bar{a})$, but then $B \models \varphi(\bar{b}, \bar{a})$.

5.3.4. Let $\Phi(\bar{x})$ be a type over a set X with respect to a structure A. Show that the following are equivalent. (a) Φ is algebraic. (b) Φ contains a formula ϕ such that $A \models \exists_{\leq n}\bar{x}\phi(\bar{x})$ for some finite n. (c) In every elementary extension of A, at most finitely many tuples realise Φ.

(a) implies (b): Suppose not. Adjoin $\neg\phi(\bar{x})$ to Φ for all algebraic formulas ϕ. By compactness, since this collection cannot have a solution, some finite set is inconsistent. Thus, Φ implies some disjunction of algebraic formulas, which is algebraic. (b) implies (a) and (c) trivially. For (c) implies (a), adjoin infinitely many tuples of constants to the language, and add to the elementary diagram of A $\Phi(\bar{c}_i)$ for every tuple $\bar{c}_i$, along with $\bar{c}_i \neq \bar{c}_j$, $(i \neq j)$. This theory is inconsistent, so by compactness, some finite piece is, so actually there is a uniform finite limit to the number of realizations of Φ, hence Φ is algebraic.

5.3.5. Let L be a first-order language and A an L-structure. Suppose X is a set of elements of A and $\bar{a}$ is a tuple of elements of A, none of which are algebraic over A. Show that some elementary extension B of A contains infinitely many pairwise disjoint tuples $\bar{a}_i$ $(i < \omega)$ which all realise $\mathrm{tp}_A(\bar{a}/X)$.

I assume the non-algebraicity of the elements is over X, not A. Let $p(\bar{x})$ be the type of $\bar{a}$ over X. Append infinitely many tuples of constants $\langle \bar{c}_i \mid i < \omega\rangle$ to the language, and consider $\mathrm{eldiag}(A) \cup \bigcup_{i<\omega} p(\bar{c}_i) \cup \bigcup_{i<j<\omega,k,l<n}\{\bar{c}_i(k) \neq \bar{c}_j(l)\}$, where $\bar{a}$ is an n-tuple, and $\bar{c}_i(k)$ is the k-th component of $\bar{c}_i$. If this theory is satisfiable, we are done. If not, by compactness, we know that there are only finitely many realizations of this type possible with all components distinct across realizations. If $n = 1$, then this means that p is algebraic. Otherwise, go by induction on n. Let B, $A \preccurlyeq B$, be a model with the maximum number of pairwise-disjoint realizations, $\bar{c}_0, \ldots, \bar{c}_{m-1}$. By induction, there is an elementary extension of B with infinitely many pairwise disjoint realizations of the type of $\bar{a}|(n-1)$ over X. Thus, any realization, $\bar{d}$, must have $\bar{d}(n-1)$ in one of the $\bar{c}_i$'s. But then the type of $\bar{a}(n-1)$ is algebraic over X in B, which is impossible.

5.3.6. Let B be an L-structure, C an elementary extension of B and X, Y sets of elements of B. (a) Prove (3.8), (3.9) and (3.10). (b) Deduce that $\mathrm{acl}_B \mathrm{acl}_B(X) = \mathrm{acl}_B(X)$.

(3.8) is $X \subseteq \mathrm{acl}_B(X)$. For each $a \in X$, the formula $x = a$ has exactly one solution. (3.9) is $Y \subseteq \mathrm{acl}_B(X)$ implies $\mathrm{acl}_B(Y) \subseteq \mathrm{acl}_B(X)$. Let $a \in \mathrm{acl}_B(Y)$. Let $B \models \varphi(a, \bar{b})$, with $\bar{b}$ in Y, and $B \models \exists_{\leq n}x\varphi(x)$, for some $n < \omega$. Since $\bar{b}$ is in $\mathrm{acl}_B(X)$, we know that for some $\psi(\bar{y})$ algebraic, $B \models \psi(\bar{b})$. Let there be m solutions to $\psi(\bar{y})$. Consider the formula $\exists\bar{y}\,(\psi(\bar{y}) \wedge \exists_{\leq n}x\varphi(x, \bar{y}) \wedge \varphi(x, \bar{y}))$. Then there are fewer than nm solutions to this formula, and a is one of them. (3.10) is "if $B \preccurlyeq C$ then $\mathrm{acl}_B(X) = acl_C(X)$." Let $a \in \mathrm{acl}_B(X)$. Then for some $\varphi(x)$ with parameters in X and some $n < \omega$, $B \models \varphi(a)$ and $B \models \exists_{=n}x\varphi(x)$. Since $B \preccurlyeq C$, both of those statements are true in C, so $a \in \mathrm{acl}_C(X)$. The other way, if $C \models \varphi(a)$ and $C \models \exists_{=n}x\varphi(x)$, then the second statement is true in B, and so there are n witnesses to it in B. If none of them are a, then C has $n+1$ witnesses, which is impossible. So a must be one of the witnesses, so $a \in \mathrm{acl}_B(X)$.

(b) In (3.9), setting $Y = \mathrm{acl}_B(X)$, we have $\mathrm{acl}_B\,\mathrm{acl}_B(X) = \mathrm{acl}_B(X)$.

5.3.7. Let A be an L-structure and X a set of elements of A. Show that there is an elementary extension B of A with a descending sequence $(C_i \mid i < \omega)$ of elementary substructures such that $\mathrm{acl}_A(X) = \bigcap_{i<\omega} \mathrm{dom}(C_i)$.

We apply Theorem 5.3.5, that if $(B, \bar{a}) \equiv (C, \bar{a})$, then there is an elementary extension D of B and an elementary embedding $g : C \to D$ with $g\bar{a} = \bar{a}$ and $\mathrm{dom}(B) \cap g(\mathrm{dom}(C)) = \mathrm{acl}_B(\bar{a})$. Apply it to $(A, \bar{a}) \equiv (A, \bar{a})$, with $\bar{a}$ a listing of X. We get $A \preccurlyeq A_1$, and a map $g_0 : A \to A_1$ with $\mathrm{dom}(A) \cap g_0(\mathrm{dom}(A)) = acl_A(X)$. Now we can look at two copies of A_1, one with A elementarily embedded in it through g_0 and one with the same copy of A an elementary substructure. Then we have $(A_1, \mathrm{dom}(A)) \equiv (A_1, g_0(\mathrm{dom}(A)))$, so (identifying $g_0(\mathrm{dom}(A))$ with $\mathrm{dom}(A)$), we can find an elementary extension A_2 of A_1 with an elementary embedding $g_1 : A_1 \to A_2$ extending g_0. Continuing this, we get a chain of models A_i. Set $B = \bigcup_{i<\omega} A_i$, and $g = \bigcup_{i<\omega} g_i$. We know that g is an elementary embedding. If $a \in A \setminus \mathrm{acl}_A(X)$, then $ga \in A_1 \setminus A$, since g_0 maps every element not in $\mathrm{acl}_A(X)$ to a new element. The same argument shows that if $a \in A_i \setminus A_{i-1}$, then $ga \in A_{i+1} \setminus A_i$. $g^i B$ is an elementary substructure of $g^{i-1}B$, since g is an elementary embedding. Let $C_i = g^i B$. Consider $C = \bigcap_{i<\omega} \mathrm{dom}(C_i)$. It is clear that $\mathrm{acl}_A(X) \subseteq C$. Assume we have $a \in C \setminus \mathrm{acl}_A(X)$. Choose k the least index such that we have $a \in (C \setminus \mathrm{acl}_A(X)) \cap A_i$, with $i = 0$ if $a \in A$. Choose such a. Then $a = g^{k+1}b$, for some $b \in B$. Suppose the least index of the A_i's containing b is l. Then by above, $l + k + 1 = k$. But this is impossible. Thus, a does not exist.

5.3.8. Let L be a first-order language and T a theory in L. Show that the following are equivalent. (a) If A is a model of T, then the intersection of any two elementary substructures of A is again an elementary substructure of A. (b) If A is a model of T, then the intersection of any family of elementary substructures of A is again an elementary substructure of A. (c) If A is a model of T and $(B_i \mid i < \gamma)$ is a descending sequence of elementary substructures of A then the intersection of the B_i is again an elementary substructure of A. (e) For any formula $\phi(x, \bar{y})$ of L there are a formula $\psi(x, \bar{y})$ of L and an integer n such that $T \vdash \forall x\bar{y}(\phi \to \exists x(\phi \wedge \psi) \wedge \exists_{\leq n} x\psi)$.

(b) to (a) and (b) to (c) are obvious. (a) to (d): Given A and X, as in the previous problem we can embed A in A_1 such that $A \cap g(A) = \mathrm{acl}_A(X)$. (c) to (d): By the previous problem, given any X, we can find a descending sequence whose intersection is $\mathrm{acl}_A(X)$. (e) to (d): We have A and X a set of elements in A, and $B = \mathrm{acl}_A(X)$. Consider $\phi(x, \bar{b})$, with $\bar{b}$ any tuple in B. We must show a solution exists in B if it does in A. Assume there is a solution in A, a. Instantiating (e) with a and $\bar{b}$, $A \models \exists x(\phi(x, \bar{b}) \wedge \psi(x, \bar{b})) \wedge \exists_{\leq n}\psi(x, \bar{b})$. Every solution to $\psi(x, \bar{b})$ is in B, and at least one of them is also a solution to $\phi(x, \bar{b})$. By Tarski-Vaught, $B \preccurlyeq A$.

We now have (d) is implied by all the others. Here is (d) to (b). For every $B_i \preccurlyeq A$, $\mathrm{acl}_{B_i}(\bigcap_{j<\gamma} B_j) = \mathrm{acl}_A(\bigcap_{j<\gamma} B_j)$. Thus, $\mathrm{acl}_A(\bigcap_{j<\gamma} B_j) = \bigcap_{i<\gamma} B_j$, so $\bigcap_{i<\gamma} B_j$ is an elementary substructure of A.

All that is left is (d) to (e). Let $\phi(x, \bar{y})$ be any formula. Then consider the sentences

$$T \cup \{\exists x \phi (x, \bar{c})\} \cup \{\neg (\exists x (\phi (x, \bar{c}) \wedge \psi (x, \bar{c})) \wedge \exists_{\leq n} x \psi (x, \bar{c})) \mid n < \omega,\, \psi \in L\},$$

with $\bar{c}$ a tuple of new constants. Suppose some finite set is inconsistent. Then $T \vdash \exists x \phi (x, \bar{c}) \rightarrow \bigvee_{i<m} (\exists x (\phi (x, \bar{c}) \wedge \psi_i (x, \bar{c})) \wedge \exists_{\leq n_i} x \psi_i (x, \bar{c}))$. Elementary rearrangement gives us $T \models \forall x \bar{y} (\phi \rightarrow (\exists x (\phi \wedge \bigvee_{i<m} \psi_i \wedge \exists_{\leq n_i} x \psi_i))$. Replacing each $\psi_i(x, \bar{y})$ by $\theta_i = \psi_i(x, \bar{y}) \wedge \exists_{\leq n_i} x \psi(x, \bar{y})$, and letting $\xi = \bigvee_{i<m} \theta_i$ and $N = \sum_{i<m} n_i$, we see that $T \vdash \forall x \bar{y} (\phi \rightarrow \exists x (\phi \wedge \xi) \wedge \exists_{\leq N} x \xi)$, so ϕ does not contradict (e). Suppose then, that the collection is consistent, so we can find a model, A. Let $\bar{c}$ interpret itself in A. Let $B = \mathrm{acl}_A(\bar{c})$. Consider $\exists x \phi(\bar{c})$ in B. If it is satisfied, then there is some witness, b, in B. Then for some ψ, $A \models \phi (b, \bar{c}) \wedge \psi(b, \bar{c})$ and $A \models \exists_{\leq n} x \psi(x, \bar{c})$ for some n. But this contradicts some statement in our collection, which is impossible. Thus, ϕ did not contradict (e). Since no ϕ does, (d) implies (e).

5.4.1. Let T be a theory in a first-order language L and $\Phi (\bar{x}) (\bar{x})$ a set of formulas of L. Show that the following are equivalent. (a) If A and B are models of T, $A \subseteq B$, $\bar{a}$ is a sequence of elements of A and $A \models \bigwedge \Phi (\bar{a})$, then $B \models \bigwedge \Phi (\bar{a})$. (b) Φ is equivalent modulto T to a set of $\exists_1$ formulas of L.

(b) easily implies (a). For the reverse, we can assume that Φ is a set of sentences, by adjoining constants $\bar{a}$. Let Ψ be the set of $\exists_1$ consequences of $T \cup \Phi$. Let B be a model of $T \cup \Psi$, and C a model of $T \cup \Phi$. Then $(C, \bar{a}) \Rrightarrow_1 (B, \bar{a})$, so there is an elementary extension D of B and an embedding g of C into D. Then, since Φ goes up, $D \models \Phi$, so $B \models \Phi$, since $B \preccurlyeq D$.

5.4.2. Let L be a first-order language and T a theory in L. Suppose A and B are models of T. Show that the following are equivalent. (a) There is a model C of T such that both A and B can be embedded in C. (b) ϕ and ψ are $\forall_1$ sentences of L such that $T \vdash \phi \vee \psi$, then either (i) A and B are both models of ϕ, or (ii) A and B are both models of ψ.

(a) implies (b): Since $C \models \phi \vee \psi$, we must have $C \models \phi$ or $C \models \psi$. Then, since ϕ and ψ are $\forall_1$, they pass down to A and B. In the other direction, consider $\mathrm{diag}\,(A) \cup \mathrm{diag}\,(B) \cup T$. Assume it is inconsistent. Then we have $T \vdash \neg\psi\,(\bar{a}) \vee \neg\theta\,(\bar{b})$, for some $\psi \in \mathrm{diag}\,(A)$, $\theta \in \mathrm{diag}\,(B)$. By the lemma on constants, $T \vdash \forall \bar{x} \neg \psi\,(\bar{x}) \vee \forall \bar{y} \neg \theta\,(\bar{y})$. But A does not satisfy the first sentence, and B does not satisfy the second, so they cannot agree.

5.4.3. Let L be a first-order language and let A_0, A_1 be L-structures with $A_0 \subseteq A_1$. Let n be a positive integer. Show that $A_0 \preccurlyeq_{2n-1} A_1$ if and only if there is a chain $A_0 \subseteq \cdots \subseteq A_{2n}$ in which $A_i \preccurlyeq A_{i+2}$ for each i: [diagram omitted] where all the arrows are inclusions. [Each A_i is included in A_{i+1}].

Suppose $A \preccurlyeq_m B$ with $m > 0$. Let $\mathrm{diag}_m\,(B)$ be the set of all $\exists_m$ formulas with parameters from B. Consider $T = \mathrm{eldiag}\,(A) \cup \mathrm{diag}_m\,(B)$. We show T is consistent. Suppose not. Then $\phi\,(\bar{a}) \vdash \neg\psi\,(\bar{a}, \bar{b})$, where $\bar{a}$ is in A and $\bar{b}$ is in $B \setminus A$, $\phi \in \mathrm{eldiag}(A)$ and $\psi \in \mathrm{diag}_m(B)$. Then $\phi(\bar{a}) \vdash \forall \bar{y} \neg \psi(\bar{a}, \bar{y})$. Thus, $A \models \neg \exists \bar{y} \psi(\bar{a}, \bar{y})$, while $B \models \exists \bar{y} \psi(\bar{a}, \bar{y})$. But ψ is $\exists_m$, so $\exists \bar{y} \psi$ is $\exists_m$, so A and B must agree on $\exists \bar{y} \psi$. Thus, $\mathrm{eldiag}(A) \cup \mathrm{diag}_m(B)$ is consistent. Let C realize it. Now consider any $\exists_{m-1}\ \varphi$, with $C \models \varphi(\bar{b})$, for $\bar{b}$ in B. If we had $B \models \neg\varphi(\bar{b})$, then since $\neg\varphi$ is $\exists_m$, we would have a contradiction. Thus, $B \models \varphi(\bar{b})$. Thus, $B \preccurlyeq_{n-1} C$. Let the existence of C given A and B be the n-existential amalgamation theorem.

Now let $A_0 \preccurlyeq_{2n-1} A_1$ be given. Applying the n-existential amalgamation theorem back and forth, we have $A_0 \subseteq \cdots \subseteq A_{2n}$, with all conditions fulfilled.

Conversely, suppose we have $A_0, \ldots, A_k$ ($k = 2n$) as above. We show by induction on n that $A_0 \preccurlyeq_{k-1} A_1$. When $k = 0$, this is is trivial. When $k = 2$, this is the fact that existential formulas are preserved in superstructures (and then in elementary substructures). When $k = i + 1$, $k > 2$, write any $\varphi(\bar{z})$ which is $\exists_k$ as $\exists \bar{x} \forall \bar{y} \theta(\bar{x}, \bar{y}, \bar{z})$, where θ is $\exists_{k-2}$. Assume $A_1 \models \varphi(\bar{a})$ for some $\bar{a}$ in A_0. Let $\bar{b}$ in A_1 witness $\bar{x}$. Then $A_1 \models \forall \bar{y} \theta(\bar{b}, \bar{y}, \bar{a})$. $A_1 \preccurlyeq A_3$, so $A_3 \models \forall \bar{y} \theta(\bar{b}, \bar{y}, \bar{a})$. Then for every $\bar{c}$ in A_2, $A_3 \models \theta(\bar{b}, \bar{c}, \bar{a})$. By induction (for $k-1$), $A_2 \preccurlyeq_{k-2} A_3$, and $\bar{b}, \bar{c}, \bar{a}$ are all in A_2. Thus, $A_2 \models \theta(\bar{a}, \bar{b}, \bar{c})$, for every $\bar{c}$ in A_2, and so $A_2 \models \forall \bar{y} \theta(\bar{a}, \bar{y}, \bar{c})$. Then, since $A_0 \preccurlyeq A_2$, $A_0 \models \exists \bar{x} \forall \bar{y} \theta(\bar{x}, \bar{y}, \bar{c})$, so $A_0 \models \varphi(\bar{a})$.

5.4.4. Let L be a first-order language, T a theory in L, n an integer ≥ 2 and $\phi(\bar{x})$ a formula of L. Show that the following are equivalent. (a) ϕ is equivalent modulo T to an $\forall_n$ formula $\psi(\bar{x})$ of L. (b) If A and B are models of T such that $A \preccurlyeq_{n-1} B$, and $\bar{a}$ is a tuple of elements of A such that $B \models \phi(\bar{a})$, then $A \models \phi(\bar{a})$. (c) ϕ is preserved in unions of $\preccurlyeq_{n-2}$ chains of models of T.

(a) implies (b): Let $\phi(\bar{x}) = \forall \bar{y} \theta(\bar{y}, \bar{x})$, with θ $\exists_{n-1}$. Then if $B \models \phi(\bar{a})$, for every choice of $\bar{c}$ in A, $B \models \theta(\bar{c}, \bar{a})$, so since $A \preccurlyeq_{n-1} B$, $A \models \theta(\bar{c}, \bar{a})$, so $A \models \forall \bar{y} \theta(\bar{y}, \bar{a})$, so $A \models \phi(\bar{a})$. For the converse, make ϕ into a sentence by appending $\bar{a}$ to the language. Let Φ be the $\forall_n$ consequences of ϕ in T. Let A be a model of $T \cup \Phi$. Consider $U = \mathrm{diag}_{\forall, n-1}(A) \cup \{\phi\} \cup T$, where the first set is all $\forall_{n-1}$ formulas with coefficients in A which A satisfies. We show U has a model. If not, then $T \vdash \phi(\bar{a}) \to \neg\psi(\bar{a}, \bar{c})$, with $\psi(\bar{a}, \bar{c})$ a $\forall_{n-1}$ formula true in A. But then $T \vdash \phi(\bar{a}) \to \forall \bar{y} \neg \psi(\bar{a}, \bar{y})$, and the last statement is $\forall_n$. Then A must satisfy it by construction, but this is impossible. Thus, U has a model, B. But then $A \preccurlyeq_{n-1} B$, and since $B \models \phi(\bar{a})$, $A \models \phi(\bar{a})$.

(a) implies (c): Suppose we have an $\preccurlyeq_{n-2}$ chain, $\langle A_i \mid i < \gamma \rangle$, with $A_i \models \phi(\bar{a})$. Let $A = \bigcup_{i<\gamma} A_i$. Suppose $A \models \neg\phi(\bar{a})$. Then, writing ϕ as above, we have $A \models \neg\theta(\bar{c}, \bar{a})$, for some $\bar{c}$ in A. But $\bar{c}$ has been added at some stage $< \gamma$, say k. $A_k \models \phi(\bar{a})$, so $A_k \models \theta(\bar{c}, \bar{a})$. Thus, since $\theta(\bar{c}, \bar{a}) = \exists \bar{z} \psi(\bar{a}, \bar{c}, \bar{z})$, we can find $\bar{d}$ in A_k such that $A_k \models \psi(\bar{a}, \bar{c}, \bar{d})$. But ψ is $\forall_{n-2}$. Thus, all we need to show is that $A_k \preccurlyeq_{n-2} A$. Claim: if $\langle B_i \mid i < \gamma \rangle$ is an $\preccurlyeq_n$-chain, then if $B = \bigcup_{i<\gamma} B_i$, $B_i \preccurlyeq_n B$. Clear for $n = 0$ (embeddings). If we know it for n, we have $B_i \preccurlyeq_n B$, and we want $B_i \preccurlyeq_{n+1} B$. Let $\varphi(\bar{a})$ be any formula with parameters from B_i with $B \models \varphi(\bar{a})$. Then $B \models \exists \bar{x} \forall \bar{y} \theta(\bar{a}, \bar{x}, \bar{y})$, for some θ $\exists_{n-1}$. Let a witness be $\bar{b}$, from B_j (if $j < i$, then just set $j = i$). Then $B \models \forall \bar{y} \theta(\bar{a}, \bar{b}, \bar{y})$. Then for all $\bar{c}$ in B_j, $B \models \theta(\bar{a}, \bar{b}, \bar{c})$. Then, by induction, $B_j \models \theta(\bar{a}, \bar{b}, \bar{c})$, so $B_j \models \forall y \theta(\bar{a}, \bar{b}, \bar{y})$, so $B_j \models \varphi(\bar{a})$, so since $B_i \preccurlyeq_n B_j$, $B_i \models \varphi(\bar{a})$. The claim is proven, and so is (a) to (c).

For (c) to (a), atomize the theory at the $\exists_{n-2}$ level. Then a $\preccurlyeq_{n-2}$ chain in the old theory is just a chain in the new theory. By the Chang-Łoś-Suszko theorem, since ϕ is preserved in unions of chains, it is $\forall_2$ in this new theory. But then since each predicate in the equivalent formula can be replaced by an $\exists_{n-2}$ formula, the equivalent formula is $\forall_n$ in the old language. The same argument works for a shorter proof of (a) implies (c).

5.4.5*. Let L be a first-order language and T a theory in L. Show that the following are equivalent. (a) T is equivalent to a set of sentences of L of the form $\forall x \exists \bar{y} \phi(x, \bar{y})$ with ϕ quantifier-free. (b) If A is an L-structure and for every element a of A there is a substructure of A which contains a and is a model of T, then A is a model of T.

(a) implies (b) easily. For (b) implies (a), T is certainly $\forall_2$, since it is preserved in unions of chains. Let A be a model of the $\forall x \exists \bar{y}$ consequences of T. We show that given any $a \in A$, there is a substructure of A containing a which is a model of T. First, pass to an L^+-saturated elementary extension of A, so we may assume that A is L^+-saturated. Let $B_0 = \langle a \rangle_A$. Suppose for some $\bar{c}$ in B_0, and some $\forall \bar{x} \exists \bar{y} \theta(\bar{x}, \bar{y})$ in T, for no $\bar{b}$ in A do we have $A \models \theta\left(\bar{c}, \bar{d}\right)$. But every element of $\bar{c}$ is a term of a, say $\bar{t}(a)$, and certainly $T \vdash \forall x \exists \bar{y} \theta(\bar{t}(x), \bar{y})$, but then A does too. Thus, no tuple in B_0 contradicts T. Now we proceed by induction: given B_i, we wish to find $B_{i+1} \subset A$ with a witness to $\theta(\bar{c}, \bar{y})$, with $\forall \bar{x} \exists \bar{y} \theta$ a sentence in T, and $\bar{c}$ a tuple in B_i. We assume that for every such θ and $\bar{c}$ in B_i, there is a witness in A. Fix θ. Let $\bar{d}$ be a witness in A. Suppose, though, that for some θ_1, and some term $\bar{t}_1\left(\bar{d}, \bar{b}\right)$, with $\bar{b}$ a tuple in B_i, $\theta_1\left(\bar{t}_1\left(\bar{d}, \bar{b}\right), \bar{z}\right)$ has no witness in A. But since $T \models \forall \bar{z} \exists \bar{w} \theta_1(\bar{z}, \bar{w})$, $T \models \forall \bar{x} \bar{u} \exists \bar{y} \exists \bar{w}\left(\theta(\bar{x}, \bar{y}) \wedge \theta_1\left(\bar{t}_1(\bar{y}, \bar{z}), \bar{u}\right)\right)$. Thus, with $\theta' = \theta \wedge \theta_1$, we can find $\bar{d}'$ which does not cause problems for θ_1 and $\bar{t}_1$. But perhaps there is a bad θ_2 now. We can repeat this process for any finite collection of θ's. Thus, $A \models \exists \bar{y}\left(\theta(\bar{c}, \bar{y}) \wedge \forall \bar{u} \bigwedge_{i<n} \exists \bar{w}_i \theta_i\left(\bar{t}_i(\bar{y}, \bar{u}), \bar{w}_i\right)\right)$, for any $n < \omega$. But this is a collection of formulas over a set of size $|B_i|$, so since A is $|L|^+$-saturated, there is a witness, $\bar{d}$, and there is a witness in A for every $\theta\left(\bar{t}\left(\bar{d}, \bar{b}\right), \bar{y}\right)$. $B_{i+1} = \langle B_i \cup \{\bar{d}\} \rangle_A$. We show now that the property that any $\theta(\bar{c}, \bar{y})$ has a witness is preserved. If not, let $\bar{c}$ be a counterexample. We can write $\bar{c}$ as $\bar{t}\left(\bar{d}, \bar{b}\right)$. But then there is a witness by choice of $\bar{d}$. Continue this process for $\gamma = |L|^+$ steps. Let $B = \bigcup_{i<\gamma} B_i$. Then every θ and $\bar{c}$ was taken care of at some point, and so B is a model of T lying in A. Thus, any element of A is contained in a substructure of A which is a model of T, so A is a model of T.

5.4.6. Let L be a first-order language and T a theory in L. Show that the following are equivalent. (a) Whenever A and B are models of T with $A \preccurlyeq B$ and $A \subseteq C \subseteq B$, then C is also a model of T. (b) Whenever A and B are models of T with $A \preccurlyeq_2 B$, and $A \subseteq C \subseteq B$, then C is also a model of T. (c) T is equivalent to a set of $\exists_2$ sentences.

(c) implies (b) implies (a) trivially. It remains to show (a) implies (c). Let C be any model of the $\exists_2$ consequences of T, U. Let A be a model of T and the $\forall_2$ consequences of $\mathrm{Th}(C)$. First we show A exists. Suppose not. Then for some φ which is $\forall_2$ in $\mathrm{Th}(C)$, $T \vdash \neg\varphi$. But since C is a model of the $\exists_2$ consequences, then $\neg\varphi$ must be true in C – contradiction. Thus, A exists. Consider $U = \mathrm{diag}_{\forall,1}(A) \cup \mathrm{Th}(C)$. We show U is consistent. If not, we have $\mathrm{Th}(C) \vdash \neg\psi(\bar{a})$, where $\psi(\bar{a})$ is $\forall_1$ and true in A. Then $\mathrm{Th}(C) \vdash \forall\bar{x}\neg\psi(\bar{x})$. Now this statement is $\forall_2$, but A satisfies every $\forall_2$ sentence in $\mathrm{Th}(C)$, so this is impossible. Thus, U has a model, C'. By existential amalgamation, we can find B, $A \preccurlyeq B$, and C' embedding into B. Then C' is a model of T, and since $\mathrm{Th}(C') = \mathrm{Th}(C)$, C is a model of T.

5.4.7. Let T be a first-order theory, and suppose that whenever a model A of T has substructures B and C which are also models of T with non-empty intersection, then $B \cap C$ is also a model of T. Show that T is equivalent to a $\forall_2$ first-order theory.

We show that T is preserved under unions of chains. Let $\langle D_i \mid i < \gamma\rangle$ be a chain of models of T, and let $D = \bigcup_{i<\gamma} D_i$. Then $\mathrm{diag}_{\forall,1}(D) \cup T$ is consistent, since every finite collection is in some large enough D_i, and $\forall$ statements go down to substructures. Thus there is a model of T, B, with $D \subseteq B$ and $B \Rrightarrow_1 D$. Take a second copy of $B \setminus D$ and adjoin it to B. Thus, we have two copies of B sharing D. Let B_2 be the copy. Now, suppose $U = T \cup \mathrm{diag}(B) \cup \mathrm{diag}(B_2) \cup \{b \neq c \mid b \in B \setminus D,\, c \in B_2 \setminus D\}$ is inconsistent. Then, by compactness, we can find $\psi\left(\bar{b}_1, \bar{d}\right)$ in $\mathrm{diag}(B)$ and $\theta\left(\bar{b}_2, \bar{d}\right)$ in $\mathrm{diag}(B_2)$ such that $T \vdash \psi\left(\bar{b}_1, \bar{d}\right) \wedge \theta\left(\bar{b}_2, \bar{d}\right) \to \bigvee_{i,j<n} \bar{b}_1(i) = \bar{b}_2(j)$, $\bar{b}_1(i)$ the i-th component of $\bar{b}_1$, and $\bar{b}_1, \bar{b}_2$ disjoint from D. Now, note that $B \models \psi\left(\bar{b}_1, \bar{d}\right) \wedge \theta\left(\bar{b}_2, \bar{b}\right)$. Thus, $B \models \exists\bar{x}\bar{y}\left(\psi\left(\bar{x}, \bar{d}\right) \wedge \theta\left(\bar{y}, \bar{d}\right)\right)$. Thus, since $B \Rrightarrow_1 D$, so does D. Let $\bar{d}_2$ be a witness for $\bar{y}$. Then $B \models \psi\left(\bar{b}_1, \bar{d}\right) \wedge \theta\left(\bar{d}_2, \bar{d}\right)$. Since $B \models T$, this means that some component of $\bar{b}_1$ must be in D, since it is equal to some component of $\bar{d}_2$, but this goes against our choice of $\bar{b}_1$. Thus, U is consistent. Let C be a model of U. Then B and B_2 are substructures of C which are models of T, so $B \cap B_2 = D$ is a model of T.

5.5.1. (Robinson's joint consistency lemma.) Let L_1 and L_2 be first-order languages and $L = L_1 \cap L_2$. Let T_1 and T_2 be consistent theories in L_1 and L_2 respectively, such that $T_1 \cap T_2$ is a complete theory in L. Show that $T_1 \cup T_2$ is consistent.

Let $B \models T_1$ and $C \models T_2$. Then $B|L \equiv C|L$. By Theorem 5.5.1, we can find D an $L_1 \cup L_2$-structure, such that $B \preccurlyeq D$ and $g : C \to D$ is an elementary embedding. Then $D \models T_1 \cup T_2$.

5.5.2. Let L and L^+ be first-order languages with $L \subseteq L^+$, and let $\phi(\bar{x})$ be a formula of L^+ and T a theory in L^+. Suppose that whenever A and B are models of T and $f : A|L \to B|L$ is a homomorphism, f preserves ϕ. Show that ϕ is equivalent modulo T to an $\exists_1^+$ formula of L.

We can make ϕ a sentence by adding new constants, $\bar{a}$. Let Φ be the $\exists_1^+$-consequences in L of ϕ in T. Let B be a model of $T \cup \Phi(\bar{a})$. Let $\Psi = \{\neg\psi \mid B \models \neg\psi(\bar{a}) \wedge \psi \in \exists_1^+ \wedge \psi \in L\}$ – the negations of all the positive existential formulas of L not true in B. Let C be a model of $T \cup \{\phi(\bar{a})\} \cup \Phi(\bar{a}) \cup \Psi(\bar{a})$. It is easy to see that this theory is consistent, so the model exists (since the disjunction of $\exists_1^+$ formulas is $\exists_1^+$). Then $(C|L, \bar{a}) \Rrightarrow_1^+ (B|L, \bar{a})$, so we can take an L-elementary extension of B, D, and an L-homomorphism from C to D. But since $D \models T|L$, we can find E, with $E \models T$, and $D \preccurlyeq E|L$. Then the

L-homomorphism from C to E shows that $E \models \phi(\bar{a})$. By Beth's definability theorem, ϕ is certainly equivalent to an L-sentence, and so, since $B|L \preccurlyeq D \preccurlyeq E|L$, $B \models \phi(\bar{a})$. Thus, ϕ is equivalent to its $\exists_1^+$-consequences in L.

5.5.3. Let L_1 and L_2 be first-order languages with $L = L_1 \cap L_2$. Suppose ϕ and ψ are sentences of L_1 and L_2 respectively, such that $\phi \vdash \psi$. Show that if every function or constant symbol of L_1 is in L_2, and ϕ is an $\forall_1$ sentence and ψ is an $\exists_1$ sentence, then there is a quantifier-free sentence θ of L such that $\phi \vdash \theta$ and $\theta \vdash \psi$.

Let Θ be the set of all quantifier-free sentences θ of L such that $\phi \vdash \theta$. Consider any L_2-structure A which is a model of Θ. Let A_0 be the substructure of A consisting only of closed terms. Note that then A_0 is an L_1-structure as well. Now consider $\{\phi\} \cup \mathrm{diag}_{L_1}(A_0)$. Suppose it is inconsistent. Then we have $\phi \vdash \neg\psi$, for $\psi \in \mathrm{diag}_{L_1}(A_0)$. But $\neg\psi$ is quantifier-free (note that $\mathrm{diag}(A_0)$ contains no new symbols, since every element of A_0 is a closed term), so this is impossible. Thus, we can find $B \models \phi$ with $A_0 \subseteq B$. Then, since ϕ is $\forall_1$, $A_0 \models \phi$, so $A_0 \models \psi$, and since ψ is $\exists_1$, $A \models \psi$, so Θ implies ψ. Then some finite subset does, so we are done.

5.5.4*. Let L_1 and L_2 be first-order languages with $L = L_1 \cap L_2$. Suppose ϕ and ψ are sentences of L_1 and L_2 respectively, such that $\phi \vdash \psi$. Show that if ϕ and ψ are both $\forall_1$ sentences then there is an $\forall_1$ sentence θ of L such that $\phi \vdash \theta$ and $\theta \vdash \psi$.

Let Θ be the set of all $\forall_1$ sentences θ of L such that $\phi \vdash \theta$. Let A_0 be any L_2-structure which is a model of Θ. Let T be the L-consequences of ϕ. $\mathrm{diag}(A_0|L) \cup T$ is consistent: suppose not, so let $T \vdash \psi(\bar{a})$, with $\psi(\bar{a}) \in \mathrm{diag}(A|L)$. Then $T \vdash \forall\bar{x}\psi(\bar{x})$. But then $\phi \vdash \forall\bar{x}\psi(\bar{x})$, so it must be true in A, which is impossible. Thus, the set is consistent, so there is a model of it, B (an L-structure). Then $B \preccurlyeq C|L$, with C a model of ϕ. Now, we can trivially extend C to be an L_2 structure containing A_0: fixing any element a in A_0, for any function symbol f, $f^C(\bar{b}) = f^{A_0}(\bar{b})$ if $\bar{b} \subseteq A_0$, and $f^C(\bar{b}) = a$ otherwise. Relations in C are the same as in A_0. Since $C \models \phi$, $C \models \psi$, so as ψ is $\forall_1$, $A_0 \models \psi$. Thus, Θ implies ψ.

5.5.5. Let L and L^+ be first-order languages with $L \subseteq L^+$, and suppose P is a 1-ary relation symbol of L^+. Let ϕ be a sentence of L^+, and T a theory in L^+ such that for every model A of T, P^A is the domain of a substructure A^* of $A|L$. Suppose that whenever A and B are models of T with $A \models \phi$ and $f : A^* \to B^*$ is a homomorphism, then $B \models \phi$. Show that ϕ is equivalent modulo T to a sentence of the form $\exists y_0 \dots y_{k-1} \left(\bigwedge_{i<k} Py_i \wedge \psi(\bar{y})\right)$, where ψ is a positive quantifier-free formula of L.

Let sentences of the above form be $\exists_1^+ P$ sentences. Let Θ consist of all $\exists_1^+ P$ consequences of ϕ. Let B be any model of $T \cup \Theta$. Let Φ consist of the negations of all $\exists_1^+ P$ sentences which are false in B. Let A be a model of $T \cup \Phi \cup \{\phi\}$. A exists by the usual argument. We show that there exists an elementary extension of $B|L$, C, such that there is a homomorphism from A^* to C^*. Consider eldiag $(B|L) \cup \text{diag}^+ (P(A))$. Assume it is not consistent. Then for some positive quantifier-free $\theta(\bar{a})$ with $\bar{a}$ in $P(A)$, and some $\varphi(\bar{b}) \in \text{eldiag}(B|L)$, $\varphi \vdash \neg\theta(\bar{a})$. But $A \models \exists\bar{x} \left(\bigwedge_{y\in\bar{x}} Py \wedge \theta(\bar{x})\right)$, so B does too, which is impossible. Thus, the collection is consistent, so we have an elementary extension, C. The interpretations of $P(A)$ in C give the homomorphism. Then C can be elementarily embedded in a model of T, D, so ϕ is true in D, hence in B.

5.5.6. Let L be the first-order language with relation symbols for 'x is a son of y', 'x is a daughter of y', 'x is a father of y', 'x is a mother of y', 'x is a grandparent of y'. Let T be a first-order theory which reports the basi biological facts about these relations (e.g. that everybody has a unique mother, nobody is both a son and a daughter, etc.). Show (a) in T, 'son of' is definable from 'father of', 'mother of', and 'daughter of', (b) in T, 'son of' is not definable from 'father of and 'mother of', (c) in T, 'father of' is definable from 'son of' and 'daughter of', (d) in T, 'mother of' is not definable from 'father of' and 'grandparent of', (e) etc. ad lib.

Write each relation by its first initial, with xSy meaning x is a son of y. Then $xSy \leftrightarrow ((yFx \vee yMx) \wedge \neg xDy)$. Consider the model of an only child with no children, a, and a father and mother, and let L be the language with just M and F. Then clearly if this model satisfies $T|L$, then we can extend it to make a a son or a daughter. Thus, S is not definable. $xFy \leftrightarrow ((ySx \vee yDx) \wedge \exists z(zFx))$. Consider the model of an only child, a father, a grandfather of the child who is not the father's father, and two children of the grandfather, neither of whom is a father. Then either one can be the mother, so mother is not definable. Grandparent is definable from mother of and father of.

5.5.7. Let L be a first-order language, L^+ the language got from L by adding one new relation symbol R, and ϕ a sentence of L^+. Suppose that every L-structure can be expanded in at most one way to a model of ϕ. Show that there is a sentence θ of L such that an L-structure A is a model of θ if and only if A can be expanded to a model of ϕ.

Let $T = \{\phi\}$. Let A and B be any models of T with $A|L = B|L$. Then $A = B$ as L^+-structures. Thus, $A \models R(\bar{a}) \leftrightarrow B \models R(\bar{a})$. Therefore, modulo ϕ, R is equivalent to a formula in L. Thus, $\phi \vdash \forall\bar{x}(R(\bar{x}) \leftrightarrow \psi(\bar{x}))$, for some formula ψ in L. Let θ be ϕ with every $R(\bar{x})$ in ϕ replaced by $\psi(\bar{x})$. Then certainly $\phi \vdash \theta$. As well, if $A \models \theta$, then the expansion $R^A = \psi(A^n)$ will make A into a model of ϕ. Thus, the models which can be expanded to models of ϕ are precisely the models of θ.

5.5.8. Let T be a first-order theory. Show that the class of groups $\{G \mid G \text{ acts faithfully on some model of } T\}$ is axiomatised by an $\forall_1$ first-order theory in the language of groups.

As stated, this problem is silly. If the action of G need not preserve any characteristics of the model, then let n be the size of the largest possible model of T. If $n \geq \omega$, then every group can act faithfully. If n is finite, every group which can be embedded in S_n will act faithfully. There are finitely many such groups, each of finite size, so they can be axiomatized by a $\forall_1$ theory, enumerating all the possibilities.

If instead G is supposed to act automorphically on models of T, we interpret T in a new language, with a relation symbol replacing each function and constant symbol, T relativized to a new unary relation symbol, A, another new unary relation symbol picking out a group, G, a new ternary relation symbol taking a group element and a structure element and mapping it to a structure element. We also add a ternary relation on the group which is the group operation, a binary relation which is the inverse, and a constant naming 1. Add axioms making this into what we want – G is a group, A is a model of T, for any $g \in G$, and any relation on A, $R(g(\bar{a})) \leftrightarrow R(\bar{a})$, and for any $g \neq h \in G$, $\exists x (Ax \wedge gx \neq hx)$. Then G is clearly a PC'_Δ class, and also clearly closed under substructures. Thus, it can be first-order axiomatized by $\forall_1$ sentences as stated, by Theorem 5.5.5.

5.5.9. Assume it has been proved that every map on the plane, with finitely many countries, can be coloured with just four colours so that two adjacent countries never have the same colour. Use the compactness theorem to show that the same holds even when the map has infinitely many countries (but each country has finitely many neighbours, of course).

Let A be any infinite map, consisting of its universe together with the symmetric antireflexive binary relation B, denoting a shared border. Let T be the theory which says that P, Q, R, S are a 4-coloring of a map. We show that if any finite subset of A can be extended to a model of T, then A can. T is $\forall_1$, so closed under substructures. Thus, if we can show that $\operatorname{diag}(A)$ is consistent with T, then we are done. But since any finite piece of A is consistent with T, the whole is, by compactness. Thus, $T \cup \operatorname{diag}(A)$ has a model, and so A has a 4-coloring.

5.5.10. An ordered group is a group whose set of elements is linearly ordered in such a way that $a < b$ implies $c \cdot a < c \cdot b$ and $a \cdot c < b \cdot c$ for all elements a, b, c. A group is orderable if it can be made into an ordered group by adding a suitable ordering. (a) Show that an orderable group can't have elements of finite order, except the identity. (b) Show from the structure theorem for finitely generated abelian groups that every finitely generated torsion-free abelian group is orderable. (c) Using the compactness theorem, show that if G is a group and every finitely generated subgroup of G is orderable, then G is orderable. (d) Deduce that an abelian group is orderable if and only if it is torsion-free.

(a) Take any element not equal to 1, a. WLOG, say $1 < a$. Then $a^i < a^{i+1}$, and so if $a^n = 1$, then $1 < a < a^2 < \cdots < a^{n-1} < 1$, which is impossible.

(b) Every finitely generated abelian group which is torsion-free must then be a direct sum of finitely many copies of $\mathbb{Z}$. Then the lexicographic ordering makes such a group ordered.

(c) If $\operatorname{diag}(G)$ is inconsistent with the order axioms, then some finite set is, so some finitely generated subgroup of G is not orderable.

(d) Any torsion-free abelian group has every finitely generated subgroup orderable, and thus is orderable. If it is not torsion-free, then by (a) it is not orderable.

5.6.1. Show that $6 \to (3)^2_2$.

Suppose each pair in $[6]^2$ is colored either red or green. There must be at least 3 numbers a, b, c such that $(0, x)$ is the same for a, b, and c. Renumbering if necessary, we can assume that $(0, 1)$, $(0, 2)$, and $(0, 3)$ all have the same color, say red. Consider the colors of $(1, 2)$, $(1, 3)$, and $(2, 3)$. If one of them is red, say $(1, 2)$, then $(0, 1, 2)$ is the desired sequence. If they are all green, then $(1, 2, 3)$ is the desired sequence.

5.6.2. Let A be an infinite structure with finite relational signature. Suppose that for every $n < \omega$, all n-element substructures of A are isomorphic. Show that there is a linear ordering $<$ such that $(\operatorname{dom}(A), <)$ is an indiscernible sequence.

Let $\langle a_i \mid i < \gamma \rangle$ be an enumeration of A. Let N be the maximum arity of any symbol in L. By Ramsey's theorem, we can find an infinite subsequence, $\langle b_i \mid i < \omega \rangle$, such that any two increasing N-tuples are isomorphic. Let B be the structure on the b_i's. Now consider T, the theory of an indiscernible linear ordering on A, along with $\operatorname{diag}(A)$. Suppose $T \cup \operatorname{diag}(A)$ is inconsistent. Then $T \vdash \neg\psi(\bar{a})$, with $\bar{c} = (c_0, \ldots, c_{m-1})$ some tuple in A and $\psi \in \operatorname{diag}(A)$. But since every tuple is isomorphic to every other tuple, just choose $\bar{b} = \left(b_{i_0}, \ldots, b_{i_{m-1}}\right)$ satisfying $\psi\left(\bar{b}\right)$. But $B \models T$, so clearly T is consistent with $\psi\left(\bar{b}\right)$, and thus $\psi(\bar{a})$. Therefore, some extension of A is a model of T. But this induces an indiscernible ordering on A.

5.6.3. We say that a tree has finite branching if every element of the tree has at most finitely many immediate successors. Prove König's tree lemma: Every tree of height ω with finite branching has an infinite branch.

Let A be a model of such a tree. Let B be a proper elementary extension, formed by adjoining a constant, ∞, and adding the statements "∞ is not a successor of a" for every $a \in A$. We now construct an infinite branch through A by following ∞. At each node which is below ∞, we choose the next node based on which of the successors lies below ∞. A unique one of them must, since in A it is true that if an element lies above a node it lies above exactly one of its successors, and there are finitely many successors, so this property is first-order. The base node lies below ∞, since A proves that every element lies above the base node. Thus, we can follow this procedure, and we never finish, because if there is a node with no successors, then it cannot lie below ∞, so we will never go to it. This creates an infinite branch.

5.6.4*. Prove that for all positive integers k, m, n there is a positive integer l such that if $[l]^k = P_0 \cup \ldots \cup P_{n-1}$ then there are $i < n$ and a set $X \subseteq l$ of cardinality at least m, such that $[X]^k \subseteq P_i$ and $|X| \geq \min(X)$.

Prove it for a fixed k, m, n. Consider the tree where each node corresponds to the expression of $[l]^k$ as $P_0 \cup \ldots \cup P_{n-1}$, for some $l \in \omega$. Some partition of $[l]^k$ automatically induces a partition of $[i]^k$ for $[i]^k$, so the nodes corresponding to this partition are ordered in the obvious way. Since, given some partition of $[l]^k$, there are only finitely many ways to extend it, we have a finitely-branching tree. It clearly has height ω. Thus, we can find an infinite branch, which defines a partition of ω. Applying Ramsey's theorem to this partition, we find an infinite set X contained in P_i, for some i. Let a be the least element in X. Take l large enough that $|\{x \in X \mid x < l\}| \geq a$. Then l works.

6.1.1. Suppose $\mathbf{K}$ is a class of L-structures, and $\mathbf{K}$ contains a structure which is embeddable in every structure in $\mathbf{K}$. Show that if $\mathbf{K}$ has the AP then $\mathbf{K}$ has the JEP too.

Let A and B be any structures in $\mathbf{K}$. Let C be the structure which can embed into any structures in $\mathbf{K}$. By AP, there is a structure D into which A and B embed, preserving the embeddings of C into A and B. But then A and B embed into some structure, so $\mathbf{K}$ has JEP.

6.1.2. Let p be a prime, and let $\mathbf{K}$ be the class of all finite fields of characteristic p. Show that $\mathbf{K}$ has HP, JEP, and AP, and that the Fraïssé limit of $\mathbf{K}$ is the algebraic closure of the prime field of characteristic p.

HP is clear (assuming the language has the inverse operation). Given A, and B and C extending A, all in $\mathbf{K}$, we can regard A, B, and C as subfields of the algebraic closure of the finite p-field, $\bar{\mathbb{F}}_p$, since they are all algebraic extensions of $\mathbb{F}_p$. Now consider the field generated by $B \cup C$ in $\bar{\mathbb{F}}_p$. It is finite, since for every element in $B \cup C$, only finitely many powers need be considered when constructing terms, so there are finitely many terms which are composed entirely of multiplication operations, and every term can be written as a sum with at most one copy of each such multiplicative term. Thus, $B \cup C$ is contained in a finite field, necessarily with characteristic p, and thus in $\mathbf{K}$. JEP follows since $\mathbb{F}_p$ is embedded in every such field. Clearly $\bar{\mathbb{F}}_p$ has age at least $\mathbf{K}$, and the argument above shows that actually it has age $\mathbf{K}$ (since every element is algebraic over $\mathbb{F}_p$). Thus, all we need show is that it is weakly homogeneous. But the theory of algebraically closed fields with characteristic p eliminates quantifiers, so if A embeds in $\bar{\mathbb{F}}_p$ as C, and $B \supseteq A$, then the quantifier-free type of A in $\bar{\mathbb{F}}_p$ actually specifies that such B exists, and so a copy of it must exist over C, so B can be mapped to that copy.

6.1.3. Let $\mathbf{K}$ be the class of finitely generated torsion-free abelian groups. Show that $\mathbf{K}$ has HP, JEP, and AP, and that the Fraïssé limit of $\mathbf{K}$ is the direct sum of countably many copies of the additive group of rationals.

HP is trivial (assuming the language has the inverse operation). By embedding any A into the direct sum of finitely many copies of $\mathbb{Q}$ (which is done easily, element by element of the generating set), we can regard B and C as both being in some $\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$. Then the group containing $B \cup C$ is finitely generated, and so amalgamates B and C over A. JEP follows since $\{0\}$ embeds into every group in $\mathbf{K}$. $\omega \times \mathbb{Q}$ certainly has age $\mathbf{K}$, so the question is whether it is weakly homogeneous. But the theory of divisible, torsion-free abelian groups eliminates quantifiers, so it is.

6.1.4. Let $\mathbf{K}$ be the class of finite boolean algebras. Show that $\mathbf{K}$ has HP, JEP, and AP, and that the

Fraïssé limit of $\mathbf{K}$ is the countable atomless boolean algebra.

HP clearly holds. Regarding models in $\mathbf{K}$ as embedded in the countable atomless boolean algebra, we can amalgamate, and JEP follows since $\{0,1\}$ embeds into every boolean algebra. The countable atomless boolean algebra has age $\mathbf{K}$ and the theory eliminates quantifiers, so it is weakly homogeneous.

6.1.5. Show that the abelian group $Z(4) \oplus \bigoplus_{i<\omega} Z(2)$ is not elementarily equivalent to any ultrahomogeneous structure.

This group, denoted G, satisfies the sentences $\exists x(x + x = 0 \wedge \exists y(y + y = x))$ and $\exists x(x + x = 0 \wedge \nexists y(y + y = x))$. Let H be any other group elementarily equivalent to G, and let a and b witness the two sentences in H. Then a and b generate subgroups, and the one defined by a can certainly be mapped onto the one defined by b, but not extended to $a/2$. Thus, H is not weakly homogeneous.

6.1.6. Let L be a finite signature with no function symbols, and $\mathbf{K}$ a class of finite L-structures which has HP, JEP and AP. Show that there is a first-order theory T in L such that (a) the countable models of T are exactly the countable ultrahomogeneous structures of age $\mathbf{K}$, and (b) every sentence in T is either $\forall_1$ or of the form $\forall\bar{x}\exists y\phi(\bar{x}, y)$ where ϕ is quantifier-free.

For every $(A, \bar{a})$ in $\mathbf{K}$ ($\bar{a}$ lists the elements of A, necessarily finite), let $\theta_{\bar{a}}(\bar{x})$ assert that $\bar{x}$ has the same quantifier-free type as $\bar{a}$. Since L is finite, θ is first-order. For every $(A, \bar{a})$ and $(B, \bar{a}b)$ in $\mathbf{K}$, we have the formula $\psi_{\bar{a},b}(\bar{x}, y)$, defined as $\theta_{\bar{a}}(\bar{x}) \to \theta_{\bar{a}b}(\bar{x}, y)$. Note that for a fixed length tuple, there are only finitely many possible θs and ψs. Let T be the collection $\forall\bar{x} \bigvee_{\bar{a}\in\mathbf{K},\, |\bar{a}|=n} \theta_{\bar{a}}(\bar{x})$, where $\bar{x}$ is an n-tuple, and $0 < n < \omega$, along with $\forall\bar{x}\exists y\psi_{\bar{a}b}(\bar{x}, y)$, for each $\bar{a}b \in \mathbf{K}$. The second set ensures that any element of $\mathbf{K}$ is in the age of any model of T, since we can build it up element by element. The first set ensures that every finitely generated substructure is in $\mathbf{K}$. Thus, the age of any model of T is precisely $\mathbf{K}$. Finally, the second set gives weak homogeneity.

6.2.1. Let X be an infinite subset of $\omega \setminus \{0\}$ such that $\omega \setminus X$ is infinite, and let Y be the set of odd positive integers. Show that player $\exists$ has a winning strategy for the game $G(\pi, X)$ if and only if she has one for $G(\pi, Y)$.

Suppose $\exists$ has a winning strategy for $G(\pi, X)$. We define how $\exists$ moves inductively, in a way that at stage $2i + 1$, $\exists$ is always faced by a position she may be faced at for some $x \in X$. Here is the strategy: given a position, find an index in X at which it may occur in $G(\pi, X)$ (we show below that this is possible). This index can be followed by at most finitely many elements of X, since $\omega \setminus X$ is infinite. Let $\exists$'s move be the conjunction of all of $\exists$'s moves at these indices. We now show that every position $\exists$ faces in $G(\pi, Y)$ is faced in $G(\pi, X)$. Clearly at stage 1 this is true. Suppose it is not true for some $i > 0$. Let $m = 2i + 1$ be the least such value. Then at stage m, $\exists$ may have a position she is never faced with in $G(\pi, X)$. However, at stage $m - 2$, $\exists$ could be faced with such a position in $G(\pi, X)$. Let a be an index where this position could occur in $G(\pi, X)$. If $a + 1 \notin X$, then clearly the position at m could occur in $G(\pi, X)$. Thus, $a+1 \in X$. But then $\exists$ made the move at $m-2$ that was the conjunction of moves at $a, a+1, \ldots, b-1$, for some $b > a$. Then the position at m could certainly be the position at the next element of X – if $\forall$ makes the same move on b that he did on $m-1$, and then makes trivial moves until the next element of X. Thus, the position at m is possible, and so $\exists$ can always move. It is easy to see that T^+ for any game played according to this strategy will be T^+ for some game played in $G(\pi, X)$, so it will have π.

For the reverse, the argument is similar, except that when $\exists$ is faced by multiple $\forall$ moves, she should consider what would happen if $\forall$ did the conjunction of those moves in $G(\pi, Y)$.

6.2.2. Show that if π_i is an enforceable property for each $i < \omega$, then the property $\bigwedge_{i<\omega} \pi_i$ (which T^+ has if it has all the properties π_i) is also enforceable.

Partition ω into infinitely many infinite sets. Assign each set to one $\exists$, in charge of a specific π_i. Since each π_i is enforceable, each $\exists$ can ensure that its π_i is satisfied, so all of them are (there is no $\forall$ here – each $\exists$ is playing against the others).

6.2.3. Let T be a complete theory in a countable first-order language L, and suppose that T has infinite models. Show that T has a countable model A such that for every tuple $\bar{a}$ of elements of A there is a formula $\phi(\bar{x})$ with $A \models \phi(\bar{a})$ such that either (a) ϕ supports a complete type over T, or (b) no principal complete type over T contains ϕ.

We repeat the proof of the omitting types theorem, except that there will be experts who assure that every n-tuple fulfills the above conditions. The construction goes as before, except that now condition $(2.4)_n$ reads (For each $n < \omega$:) Ensure that for every n-tuple $\bar{c}$ of distinct witnesses, either $\psi(\bar{c})$, where ψ is not in any principal type, or $\varphi(\bar{c})$, where φ generates some principal type. It remains to be shown that expert E_n can do this. She lists all n-tuples of distinct elements, $\langle \bar{c}_i \mid i \in Y \rangle$, where Y is her subset of ω. She is given T_{i-1}, a finite set of sentences, with $i \in Y$. If $T \cup T_{i-1}$ is consistent with $\varphi(\bar{c}_i)$, for some $\varphi \in L$ generating a principal type of T, she sets $T_i = T_{i-1} \cup \{\varphi(\bar{c}_i)\}$. If not, she writes $\bigwedge T_{i-1}$ as a sentence $\chi(\bar{c}_i, \bar{d})$, where $\chi(\bar{x}, \bar{y})$ is in L, and $\bar{d}$ lists the distinct witnesses which occur in T_{i-1} but not in $\bar{c}_i$. Then $T \cup \{\varphi(\bar{c}_i)\} \vdash \neg\chi(\bar{c}_i, \bar{d})$, for every φ generating a principal type. By the lemma on constants, $T \cup \{\varphi(\bar{c}_i)\} \vdash \forall \bar{y} \neg\chi(\bar{c}_i, \bar{y})$. Then no principal type contains the formula $\exists \bar{y} \chi(\bar{x}, \bar{y})$, but $T \cup T_i \vdash \exists \bar{y} \chi(\bar{x}, \bar{y})$, so set $T_i = T_{i-1}$.

At the end of this procedure, every distinct n-tuple (and hence every n-tuple) $\bar{c}$ will satisfy either a principal type or a formula which is in no principal type. Thus, the canonical model formed from the c's is the desired one.

6.2.4. (a) Let A be a finite or countable structure of countable signature. Show that A is atomic if and only if A is a prime model of $\mathrm{Th}(A)$. (b) Deduce that any two prime models of a countable complete theory are isomorphic.

(a) If A is atomic, A is easily embeddable in any model of $\mathrm{Th}(A)$, just by enumerating all of A as $\langle a_i \mid i < \omega \rangle$, and then mapping a_n to the realization of $\mathrm{tp}(a_n/(a_0, \ldots, a_{n-1}))$ over the image of $(a_0, \ldots, a_{n-1})$, which is possible since every type is principal, and thus realized in every model. Inversely, assume A is not atomic, so it has some tuple $\bar{a}$ whose type is non-principal. By the omitting types theorem, find a model omitting this type. Then $\bar{a}$ cannot be mapped into this model, so A is not prime.

(b) By Theorem 6.2.3, if A and B are atomic, they are back-and-forth equivalent. Thus, if they are countable, they are isomorphic. Any prime model of a countable theory is countable, and by (a) is thus atomic. Thus, any two prime models are isomorphic.

6.2.5. (a) Show that if A and B are elementarily equivalent ω-saturated structures then A is back-and-forth equivalent to B. (b) Show that if A is ω-saturated and B is a countable structure elementarily equivalent to A, then B is elementarily embeddable in A.

(a) Given a position $(\bar{a}, \bar{b})$, with $(A, \bar{a}) \equiv (B, \bar{b})$, and any choice $c \in A$, choose an element d in B such that $(A, \bar{a}c) \equiv (B, \bar{b}d)$. Then certainly $(A, \bar{a}c) \equiv_0 (B, \bar{b}d)$.

(b) Play a back-and-forth game where $\forall$ chooses only from B, to exhaustion. $\exists$ can always match the move, by ω-saturation. In the end, $(B, \bar{b}) \equiv (A, \bar{a})$ where $\bar{b}$ lists the elements of B, and $\bar{a}$ are $\exists$'s moves, since any formula mentions only finitely many elements. Then B embeds in A as $\bar{a}$.

6.2.6. Show that the following are equivalent, for any countable complete first-order theory T with infinite models. (a) T has a countable ω-saturated model. (b) T has a countable model A such that every countable model of T is elementarily embeddable in A. (c) For every $n < \omega$, $S_n(T)$ is at most countable.

(a) implies (b) and (c) trivially. (b) implies (c) since any model realizes only countably many types, so if $S_n(T)$ was not countable for some n, take a model realizing an n-type not in the model specified by (b). We can shrink this model to be countable. Then it is not embeddable in the model of (b). (c) implies (a) since we can realize all types: let A be any countable model of T. List the n-types of A over every tuple $\bar{a}$ as $\langle p_i(\bar{x}, \bar{a}_i) \mid i < \omega \rangle$ (there are countably many because otherwise for some n and some m the length of $\bar{a}$, there are uncountably many $m + n$-types). We can realize p_0 over A in some countable elementary extension. Then realize the next consistent type over that. After ω steps, we have realized all consistent types over every $\bar{a}$ in A, and we are still countable. Now perform this operation ω times, and let B be the resulting model. We are still countable, and now if $(B, \bar{b}) \equiv (C, \bar{c})$, and $d \in C$, consider $p(x, \bar{c}) = \mathrm{tp}(d/\bar{c})$. We know that $\bar{b}$ came at some finite stage, so at some point, we had $p(x, \bar{b}) = p_i(x, \bar{a}_i)$, for some i. If a realization was found, then we are done. If no realization was found, then $p(x, \bar{b})$ was inconsistent, and so for some $\psi(x, \bar{b}) \in p(x, \bar{b})$, we have $B \models \forall x \neg\psi(x, \bar{b})$. But since $(B, \bar{b}) \equiv (C, \bar{c})$, $C \models \forall x \neg\psi(x, \bar{c})$, which is impossible. So a realization was found.

6.2.7. Give an example of a countable first-order theory which has a countable prime model but no countable ω-saturated model.

Let T be the theory of the rationals as a linear ordering, with every element named. Then the prime model is just the rationals, but since there are uncountably many 1-types, there is no countable ω-saturated model.

6.2.8*. Let L be a countable first-order language, T a theory in L, and $\Phi(\bar{x})$ and $\Psi(\bar{y})$ sets of formulas of L. Show that (a) implies (b): (a) for every formula $\sigma(\bar{x},\bar{y})$ of L there is a formula $\psi(\bar{y})$ in Ψ such that for all $\phi_1(\bar{x}),\ldots,\phi_n(\bar{x})$ in Φ, if $T\cup\{\exists\bar{x}\bar{y}(\sigma\wedge\phi_1\wedge\ldots\wedge\phi_n)\}$ has a model then $T\cup\{\exists\bar{x}\bar{y}(\sigma\wedge\phi_1\wedge\ldots\wedge\phi_n\wedge\neg\psi)\}$ has a model; (b) T has a model which realises Φ and omits Ψ.

If $\Phi = \{x \neq x\}$ and $\Psi = \{x \neq x\}$, then (a) holds but $T \cup \Phi$ is inconsistent, so the statement is incorrect. Assume then that $T\cup\Phi$ is consistent. We build a model in the same way as with the omitting types theorem, except that now we start with a tuple $\bar{a}$ with $\Phi(\bar{a})$, and (2.4) is just one task, as follows. We are at stage i, given T_{i-1}, consistent with $T \cup \Phi(\bar{a})$, and have a listing of tuples, $\langle \bar{c}_j \mid j < \omega\rangle$. Write T_{i-1} as a sentence, $\chi(\bar{c}_i,\bar{a},\bar{d})$. Let $\sigma(\bar{x},\bar{y}) = \exists\bar{z}\chi(\bar{x},\bar{y},\bar{z})$. We show that $T\cup\Phi(\bar{a})\cup\{\sigma(\bar{c}_i,\bar{a}),\psi\}$, where ψ is the associated formula for σ from (a), is consistent. Assume not. Then we can choose a finite set, $\phi_1,\ldots,\phi_n$ in Φ by compactness such that $T\cup\{\phi_1(\bar{a}),\ldots,\phi_n(\bar{a}),\sigma(\bar{c}_i,\bar{a})\} \vdash \psi$. We know that $T\cup\{\exists\bar{x}\bar{y}(\sigma\wedge\phi_1\wedge\ldots\wedge\phi_n)\}$ has a model, since $T_{i-1}\cup T\cup\Phi(\bar{a})$ has a model. But then by (a) $T\cup\{\exists\bar{x}\bar{y}(\sigma\wedge\phi_1\wedge\ldots\wedge\phi_n\wedge\neg\psi)\}$ has a model, contradicting the proof of ψ from σ and the ϕ_i's. Thus, the assumption was wrong, so we can set $T_i = T_{i-1}\cup\{\neg\psi(\bar{c}_i)\}$. The resulting theory's canonical model will omit Ψ and realize Φ.

6.2.9. Show that the theory T of Example 1 is complete as follows. If A is a model of T and s is a subset of ω, $|\Phi_s(A)|$ is the number of elements of A which realise Φ_s. (a) Show that if A is a model of T, then A is determined up to isomorphism by the cardinals $|\Phi_s(A)|$ ($s \subseteq \omega$). (b) Show that if $s \subseteq \omega$ and A is a model of T of cardinality $\leq 2^\omega$, then A has an elementary extension B of cardinality 2^ω with $|\Phi_s(B)| = 2^\omega$. (c) By iterating (b), show that every model of T of cardinality $\leq 2^\omega$ has an elementary extension C of cardinality 2^ω with $|\Phi_s(C)| = 2^\omega$ for each $s \subseteq \omega$.

(a) Suppose A and B are two models of T with the same values for $|\Phi_s(-)|$, for all $s \subseteq \omega$. Then just map every element in $\Phi_s(A)$ onto $\Phi_s(B)$, for each $s \subseteq \omega$. This is an embedding, since every unary relation is preserved, and onto, thus an isomorphism.

(b) Given a model A, T asserts that, for any n, and any finite subsets d of s and e of $\omega \setminus s$, there are $> n$ elements in $\bigwedge_{i\in d} P_i(x) \wedge \bigwedge_{i\notin d} P_i(x)$. By compactness, if we add 2^ω many constants to the language, and the statements that they are all in Φ_s, then eldiag(A) along with this new theory is satisfiable in some elementary extension of A, with size 2^ω, since that is the size of the language.

(c) Iterating (b), we add $2^\omega \cdot 2^\omega$ constants, so the language still has size 2^ω, and we have an elementary extension in which every class has size 2^ω. Then, given A and B any models of size $\leq 2^\omega$, $A \preccurlyeq C$ for some model of this kind, and $B \preccurlyeq D$, but $C \cong D$, so $A \equiv C \equiv D \equiv B$.

6.3.1. Show that if A is an infinite structure and $\bar{a}$ is a tuple of elements of A, then $\mathrm{Th}(A, \bar{a})$ is ω-categorical if and only if $\mathrm{Th}(A)$ is ω-categorical.

Suppose for some n, $S_n(\mathrm{Th}(A, \bar{a}))$ is infinite. Then, if m is the length of $\bar{a}$, the number of $m+n$-types over $\mathrm{Th}(A)$ is infinite, since every n-type extends to an $m+n$-type where the type of the first m variables is that of $\bar{a}$, so $\mathrm{Th}(A)$ is not ω-categorical. Inversely, if $S_n(\mathrm{Th}(A, \bar{a}))$ is finite for every n, then since every n-type of $\mathrm{Th}(A)$ is contained in at least one n-type of $\mathrm{Th}(A, \bar{a})$, $S_n(\mathrm{Th}(A))$ is finite.

6.3.2. Show that if A is a countable structure, then A is ω-categorical if and only if for every tuple of elements of A, the number of orbits of $\mathrm{Aut}(A, \bar{a})$ on single elements of A is finite.

If A is ω-categorical, then $(A, \bar{a})$ is ω-categorical for any tuple $\bar{a}$, so there are only finitely many 1-types over $\bar{a}$, so the number of orbits of $\mathrm{Aut}(A, \bar{a})$ is finite. Conversely, if the number of orbits of $\mathrm{Aut}(A, \bar{a})$ is finite, there are only finitely many 1-types realized over $\bar{a}$. We show that there are only finitely many n-types for every n realized over $\emptyset$, by induction. Suppose we have the result for n. Let $\bar{c}_0, \ldots, \bar{c}_m$ be the realizations of the n-types. Then $\mathrm{Aut}(A, \bar{c}_i)$ has finitely many orbits, so there are finitely many 1-types over $\bar{c}_i$, and thus finitely many $n+1$-types, since an $n+1$-type can be decomposed into an n-type and a 1-type over that. Then $\mathrm{Aut}(A)$ has finitely many orbits for each n, so A is ω-categorical.

6.3.3*. Show that if T is a first-order theory with countable models, then T is ω-categorical if and only if all the models of T are pairwise back-and-forth equivalent.

For the reverse, if A and B are countable models of T, then they are back-and-forth equivalent, then since they are countable, they are isomorphic. Since this is true for all countable models, all countable models of T are isomorphic, so T is ω-categorical.

The statement is not true in the forwards direction. Let L be a language with ω_1 constants, $\langle c_i \mid i < \omega \rangle$. Let T be the theory which says that if $c_0 = c_1$, then every constant is equal to c_0, and if $c_0 \neq c_1$, then every constant is distinct. (It has sentences of the form $c_0 = c_1 \rightarrow c_i = c_0$ and $c_i = c_j \rightarrow c_0 = c_1$ for $i, j < \omega_1$.) As well, let T say that there are infinitely many elements not equal to c_0. Then T has precisely 1 countable model, but clearly that model is not back-and-forth equivalent to the model in which every c_i is distinct. I thus assume that L is countable.

If T is ω-categorical and L is countable, then T is necessarily complete. $S_n(T)$ is finite for every n, and every type is realized in every model. Let A and B be any models of T. We give a strategy for $\exists$ that preserves $(A, \bar{a}) \equiv (B, \bar{b})$. Note that if the tuples have length 0, then $A \equiv B$, since T is complete. Suppose $\bar{a}$ and $\bar{b}$ have been chosen, with the desired property. Let $\forall$ choose c from A. Passing to a countable elementary substructure of A, A', containing $\bar{a}c$ and a countable elementary substructure of B, B', containing $\bar{b}$, we can find d in B' such that $(A', \bar{a}c) \equiv (B', \bar{b}d)$, and passing upwards preserves the equivalence, since the substructures are elementary.

6.3.4. Let T be a complete and countable first-order theory. Show that T is ω-categorical if and only if every countable model of T is atomic.

If T is ω-categorical, then every type in T is principal, and so every model realizes only principal types. Thus, the unique countable model realizes only principal types, and thus is atomic. Any two countable atomic models are easily isomorphic, by back-and-forth (Theorem 6.2.3) coupled with countability. Thus, T is ω-categorical.

6.3.5. Give an example of an ω-categorical first-order theory T such that no skolemisation of T is ω-categorical.

Consider the theory of dense linear orders without endpoints. Then any skolemization must include a function $f(x)$ such that $x < f(x)$. Now there are infinitely many 2-types, with $p_n(x, y)$ containing the formula $f^n(x) < y < f^{n+1}(x)$, for each n. Thus, since there are infinitely many 2-types, this skolemization cannot be ω-categorical.

6.3.6. Let B be a countable boolean algebra. Show that B is ω-categorical if and only if B has finitely many atoms.

Let B have infinitely many atoms. Consider the set of formulas which say 'x contains at least n atoms,' for each $n < \omega$. By compactness, this set can be extended to a type. Assume B is ω-categorical, so there are only finitely many possible extensions, all principal. Let $\varphi_0, \ldots, \varphi_{n-1}$ generate these types. Then the above collection implies $\bigvee_{i<n} \varphi(x)$. But then some finite subset does, so one of the φ_i's is true given only finitely many of the above formulas, and so does not generate the type. Thus, our assumption was wrong, and B is not ω-categorical.

Inversely, let B have finitely many atoms. I claim that for any D with $D \models \mathrm{Th}(B)$, B and D are back-and-forth equivalent. Certainly $B \equiv_0 D$. I show that this is a winning position for $\exists$. For a single element, these are the characteristics we look at: (i) how many atoms are contained in the element, (ii) if it contains the complement of the union of the atoms (C), and if not, (iii) if it intersects C. Now suppose we have played the game to $(B, \bar{b})$ and $(D, \bar{d})$, with the above 3 properties the same for every element of the boolean algebras generated by $\bar{b}$ and $\bar{d}$ (denoted $\mathcal{B}(\bar{b})$ and $\mathcal{B}(\bar{d})$ respectively). Clearly $(B, \bar{b}) \equiv_0 (D, \bar{d})$. We show that we can extend these sequences one more. Let $\forall$ choose $a \in B$. Now, each atom in $\mathcal{B}(\bar{b}a)$ is, for some Q an atom in $\mathcal{B}(\bar{b})$, either $Q \cap a$ or $Q \cap a^*$. For each Q an atom in $\mathcal{B}(\bar{b})$, let R be the corresponding atom in $\mathcal{B}(\bar{d})$. Let $Q \cap a$ contain k atoms, and $Q \cap a^*$ contain m atoms. Then Q contains $k+m$ atoms, and so R contains $k+m$ atoms as well. Let c' be the union of k of them. If $Q \cap a$ contains C, then Q does, and thus R does. Let $c'' = c' \cup C$. If $Q \cap a \cap C$ is non-empty, but $Q \cap a^* \cap C$ is empty, let $c'' = c' \cup (R \cap C)$. If $Q \cap a \cap C$ is empty, let $c'' = c$. Otherwise, if $Q \cap a \cap C$ and $Q \cap a^* \cap C$ are both non-empty, then since C has no atoms in it, and $R \cap C$ is non-empty, we can find $E \subset R \cap C$ such that $E^* \cap R \cap C \neq 0$. Set $c'' = c' \cup E$. Now, let c be the union of all the c'' elements constructed in this way. It is easy to see that $(B, \bar{b}, a)$ and $(D, \bar{d}, c)$ have the above 3 properties the same for every element of the boolean algebras. Thus, $\exists$ can win the back-and-forth game, and so D and B (if countable) are isomorphic. Thus, B is ω-categorical.

6.4.1. Show that if M is a countable ω-categorical structure, then there is a definitional expansion of M which is ultrahomogeneous.

We can atomize $\mathrm{Th}(M)$, so that it is quantifier-free. Let M' be the definitional expansion of M from this. Then if A and B are two finitely-generated substructure of M' and there is an isomorphism of A onto B, then letting $\bar{a}$ be a tuple of generators of A, the image of $\bar{a}$, $\bar{b}$, generates B and moreover $\bar{a}$ and $\bar{b}$ have the same type (not just quantifier-free type) over M. But then there is an automorphism of M taking $\bar{a}$ to $\bar{b}$, by Corollary 6.3.3. Since M' is a definitional expansion, this automorphism extends to M'.

6.4.2. Let L be the first-order language whose signature consists of one 2-ary relation symbol R. For each integer $n \geq 2$, let A_n be the L-structure with domain n, such that $A_n \models \neg R(i,j)$ iff $i+1 \equiv j \pmod{n}$. If S is any infinite subset of ω, we write $\mathbf{J}_S$ for the class $\{A_n \mid n \in S\}$ and $\mathbf{K}_S$ for the class of all finite L-structure C such that no structure in $\mathbf{J}_S$ is embeddable in C. Show (a) for each infinite $S \subseteq \omega$, the class $\mathbf{K}_S$ has HP, JEP, and AP, and is uniformly locally finite, (b) if S and S' are distinct infinite subsets of ω then the Fraïssé limits of $\mathbf{K}_S$ and $\mathbf{K}_{S'}$ are non-isomorphic countable ω-categorical L-structures, (c) if L is the signature with one binary relation symbol, there are continuum many non-isomorphic ultrahomogeneous ω-categorical structures of signature L.

(a) Let an "n-cycle" be a set of n elements, $a_1, \ldots, a_n$, such that $\neg R(a_i, a_i + 1)$, for $i < n$, and $\neg R(a_n, a_1)$, and for every other pair, R holds. Note that an L-structure C is in $\mathbf{K}_S$ if it has no n-cycles for $n \in S$. Clearly HP holds. If A and B are finite structures in $\mathbf{K}_S$, first map B to a structure which has no elements in common with A, B' (if $m = \max(A)$, add m to every element of B). Now the union of A and B' still has no n-cycles for $n \in S$, so we have JEP. Finally, if B and C both contain A and have no n-cycles, again take images of B and C so that $B \setminus A$ and $C \setminus A$ have no elements in common. Then consider their union, D. Any n-cycle must contain elements of $B \setminus A$ and $C \setminus A$, but for any a, b in an n-cycle, either $R(a,b)$ or $R(b,a)$, and this cannot be true, since $\neg R(a,b)$ for all $a \in B \setminus A$ and $b \in C \setminus A$. Thus, AP holds. $\mathbf{K}_S$ is uniformly locally finite since there are no function or constant symbols.

(b) WLOG, let $n \in S \setminus S'$. Then the Fraïssé limit of $\mathbf{K}_S$ contains no n-cycle, while the Fraïssé limit of $\mathbf{K}_{S'}$ does, since an n-cycle is a finitely-generated structure into which no element of $\mathbf{J}_{S'}$ can be embedded. It is clear that any isomorphism preserves n-cycles, so $\mathbf{K}_S$ and $\mathbf{K}_{S'}$ are not isomorphic. They are countable since they are Fraïssé limits, and ω-categorical since Theorem 6.4.1 applies.

(c) Since there are continuum many distinct infinite sets S, and each gives a different countable ω-categorical L-structure, there are continuum many distinct ω-categorical L-structures in the language with one binary relation.

6.4.3. We define a graph A on the set of vertices $\{c_i \mid i < \omega\}$ as follows. When $i < j$, first write j as a sum of distinct powers of 2, and then make c_i adjacent to c_j iff 2^i occurs in this sum. Show that A is the random graph.

We show that if X and Y are disjoint finite sets of vertices in A, then there is an element not in the union which is adjacent to all vertices in X and none in Y. This suffices, by Theorem 6.4.4. Let $a_1, \ldots, a_n$ be the indices of the vertices in X, and $b_1, \ldots, b_m$ those in Y. Let $m = \max(b_1, \ldots, b_m)$. Consider the vertex whose index is $2^{a_1} + \ldots 2^{a_n} + 2^{m+1}$. Then for each $i \leq n$, this vertex connects to every vertex indexed by a_i, but for each $i \leq m$, not to any indexed by b_i, since it has index greater than the index of b_i, and yet does not have 2^{b_i} in its expansion.

6.4.4. Show that if Γ is the random graph, then the vertices of Γ can be listed as $\{v_i \mid i < \omega\}$ in such a way that for each i, v_i is joined to v_{i+1}.

Let $\langle a_i \mid i < \omega\rangle$ be any enumeration of the random graph. We build the desired enumeration $\langle v_i \mid i < \omega\rangle$ by induction. Set $v_0 = a_0$. Given $v_0, \ldots, v_i$, let a be the first element in our enumeration of Γ not in this list. If a is connected to v_i, set $v_{i+1} = a$. Otherwise, choose v_{i+1} by setting $X = \{v_0, \ldots, v_i, a\}$, and $Y = \emptyset$, and applying Theorem 6.4.4. Then at the next stage, we will have $v_{i+2} = a$. In this way, every element is enumerated, and the desired property holds.

6.4.5. Show that if the set of vertices of the random graph Γ is partitioned into finitely many sets X_i $(i < n)$, then there is some $i < n$ such that the restriction of Γ to X_i is isomorphic to Γ.

We go through the X_i's one by one. If X_i is not the random graph, then we can find Y_i and Z_i, finite and disjoint, such that every element in X_i is not connected to some element in Y_i or is connected to some element in Z_i. Now let $Y = \bigcup_{i<n} Y_i$ and $Z = \bigcup_{i<n} Z_i$. Since Γ is the random graph, we can find some x disjoint from Z and connected to Y. But x cannot be in any X_i – contradiction. Thus, some X_i is the random graph.

6.4.6. Let n be an integer ≥ 3 and let $\mathbf{K}_n$ be the class of finite graphs which do not have K_n as a subgraph. (a) Show that $\mathbf{K}_n$ has HP, JEP and AP. (b) If Γ_n is the Fraïssé limit of $\mathbf{K}_n$, show that Γ_n is the unique countable graph with the following two properties: (i) every finite subgraph of Γ_n is in $\mathbf{K}_n$, and (ii) if X and Y are disjoint finite sets of vertices of Γ_n, and K_{n-1} is not embeddable in the restriction of Γ_n to X, then there is a vertex in Γ_n which is joined to every vertex in X and to no vertex in Y.

(a) HP is obvious. For JEP, just take the union. Any embedding of K_n into the union must lie entirely on one side, since any two points from different structures are disconnected. For AP, embed the two graphs so that they have no points in common besides the base structure, and so that there are no new connections. Then, again, an embedding of K_n must lie entirely on one side.

(b) Γ_n certainly has (i). If X and Y are as in (ii) for Γ_n, then consider $X \cup Y$ with the new point v, connected to every point in X and to no point in Y. This graph is certainly in $\mathbf{K}_n$, since K_n was not embeddable in $X \cup Y$, and if it is embeddable in $X \cup Y \cup \{v\}$, then K_{n-1} must be embeddable in X. Thus, it is in Γ_n. Any A with (i) has age at most $\mathbf{K}_n$. Suppose the age of A is not $\mathbf{K}_n$. Let G be the smallest (by number of vertices) graph in $\mathbf{K}_n$ which is not a substructure of A (thus non-empty). Take g any element of G. Let X be the vertices connected to g in G, and Y the vertices not connected to g. Since $G \in \mathbf{K}_n$, K_{n-1} does not embed into X. By minimality of G, $X \cup Y$ can be embedded into A. But then A fails (ii). So the age of A is $\mathbf{K}_n$. We now show A is weakly homogeneous, which proves that it is isomorphic to Γ_n. Let $B \in \mathbf{K}_n$ be a finite graph embedding into A, and let $C \in K_n$ extend B. We need only show the result if $|C \setminus B| = 1$. Let X be the set of vertices in B which are connected to the extra element of C, and Y the vertices which are not. Then by (ii) for A, we can find an element in A which is connected to those in the image of X, and not in the image of Y, so we can extend the embedding to the extra element of C, and hence on and on. Thus, A is weakly homogeneous.

6.4.7. Show that the zero-one law (Theorem 6.4.6) still holds if the signature L is empty.

Suppose L is empty, and let φ be a sentence. Let n be the number of quantifiers in φ. Suppose there is a structure A such that $A \models \varphi$ and $|A| > n$. Let C be any L-structure with $|C| > |A|$. Suppose $C \models \neg\varphi$. Play a back-and-forth game. $\forall$ chooses from whichever structure there is an $\exists$ quantifier outermost. $\exists$ makes the choice to preserve the quantifier-free type, which is just whether or not elements are equal. Since there are more than n elements, and only n choices to be made, $\exists$ can always do this. But at the end of the game, we have tuples which have the same quantifier-free type but differ on a quantifier-free formula. This is impossible, so $C \models \varphi$. Thus, if there is a structure A with $|A| > n$ and $A \models \varphi$, then φ is true in all structures with size $\geq |A|$, and thus $\lim_{n\to\infty} \mu_n(\phi) = 1$. Otherwise, considering φ's negation, it is clear that the limit is 0.

6.4.8. Let L be the first-order language whose signature consists of one 1-ary function symbol F. Show that $\lim_{n\to\infty} \mu_n(\forall x F(x) \neq x) = 1/e$.

The chances that, in a structure with n elements, any element will be mapped to itself is $1/n$. The chance that no element will be mapped to itself is $(n-1)^n/n^n$. The limit of this expression as n goes to infinity is $1/e$.

6.4.9. Prove that if $0 < k < 1$ and m is a positive integer, then $n^m \cdot k^n \to 0$ as $n \to \infty$.

Let $k = 1/(1+p)$. If $n > 2(m+1)$, then consider the binomial expansion of $(1+p)^n$. The $(m+1)$-th term is $n!p^{m+1}/(m+1)!(n-m-1)!$. Since $n > 2(m+1)$, $n!/(n-m-1)! > (n/2)^{m+1}$, so $(1+p)^n > (n/2)^{m+1}p^{m+1}/(m+1)!$. So $n^m k^n < (n^m(m+1)!)/((n/2)^{m+1}p^{m+1}) = (2^{m+1}(m+1)!/p^{m+1})/n$, and therefore goes to 0 as $n \to \infty$.

7.1.1*. Let L be a signature with just finitely many symbols, and $\mathbf{J}$ an at most countable set of finite (NB: not just finitely generated) L-structures which has HP, JEP, and AP. Let $\mathbf{K}$ be the class of all

L-structures whose age is $\subseteq \mathbf{J}$. If C is a finite or countable structure in $\mathbf{K}$, show that C is existentially closed in $\mathbf{K}$ if and only if C is the Fraïssé limit of $\mathbf{J}$.

The Fraïssé limit is e.c. in $\mathbf{K}$ by Example 2. We show the converse, that if a structure is e.c. in $\mathbf{K}$, then it is weakly homogeneous (and thus ultrahomogeneous). Let C be e.c. in $\mathbf{K}$ with age $\mathbf{J}$. Let D be the Fraïssé limit of $\mathbf{J}$. We are given $A \subset B \subset C$, and an embedding $f : A \to C$, and we wish to extend f to B. List the elements of A as $\bar{a}$, and let $B \setminus A = \bar{b}$. Since every finitely generated structure in C is actually finite, there are only finitely many distinct elements in C of the form $t(\bar{a}\bar{b})$, where t is a term of L. Thus, the complete quantifier-free type of $\bar{b}$ over $\bar{a}$ need involve only finitely many terms. Then, since L is finite, the complete quantifier-free type of $\bar{b}$ over $\bar{a}$ is a single formula, $\theta(\bar{x}, \bar{a})$. Now, C embeds in D, and thus A embeds in D, say through g. Therefore, we can extend g to B. Thus, $D \models \exists\bar{x}\theta(\bar{x}, g(f(\bar{a})))$. Taking $g(C)$, the isomorphic image of C, this means that we can find $\bar{b}' \in g(C)$ with $g(C) \models \theta(\bar{b}', g(\bar{a}))$. Then $f(\bar{b}) = g^{-1}(\bar{b}')$ extends f.

7.1.2. Show that there is a unique countable e.c. linear ordering, namely the ordering of the rationals.

Clearly, any e.c. linear ordering is dense and has no endpoints. But any countable dense linear ordering without endpoints is isomorphic to the rationals by a back-and-forth argument.

7.1.3. Show (a) an e.c. integral domain is the same thing as an e.c. field, (b) an e.c. field is never an e.c. commutative ring.

(a) The formula $\exists y(xy = 1)$ ensures that any e.c. integral domain is a field (and is realized in the field of fractions of the integral domain). Then, since the class of integral domains contains the class of fields, a structure which is e.c. in the class of integral domains must be e.c. in the class of fields. If we have an e.c. field contained in an integral domain, we can extend the integral domain to an e.c. integral domain, which is a field. Thus, considering integral domains adds no new $\exists_1$ formulas that the e.c. field needs to satisfy.

(b) The formula $\exists y(y \neq 0 \wedge y^2 = 0)$ ensures that no e.c. field is an e.c. commutative ring. If K is the e.c. field, $K[x]/(x^2)$ extends K to a commutative ring satisfying this statement.

7.1.4. Show that if A is an e.c. abelian group, then A is divisible and has infinitely many p-ary direct summands for each prime p.

The formulas $\varphi_n(x)$, $\exists y(ny = x)$ ensures that any e.c. abelian group is divisible. We show they are satisfiable. Assume that for some G and some $a \in G$, $0 < n < \omega$, it is not. Let $\varphi = \varphi_n$. Let T be the theory of abelian groups. Then $T \cup \mathrm{diag}(G) \cup \varphi(a)$ is inconsistent, so let $\psi(\bar{g}, a) \in \mathrm{diag}(G)$ be such that $T \vdash \psi(\bar{g}, a) \to \neg\varphi(a)$. Then take the abelian group generated by $\bar{g}a$, H. By the structure theorem for finitely generated abelian groups, we can write H as a direct sum of cyclic groups, $\mathbb{Z}/n_i\mathbb{Z}$ and a torsion-free component, some number of copies of $\mathbb{Z}$. Then H easily embeds into the direct sum of cyclic groups $\mathbb{Z}/(n \cdot n_i)\mathbb{Z}$ and a torsion-free component, with copies of $\mathbb{Q}$ instead of $\mathbb{Z}$, but this group clearly makes every element in H divisible by n. Therefore $T \cup \{\psi(\bar{g}, a), \varphi(a)\}$ is consistent, contradiction.

We can always adjoin the statement $\exists x(px = 0 \wedge \bigwedge_{i<n} x \neq c_i)$ for each c_i with order p that we have so far (just take the group and direct-sum it with $\mathbb{Z}/p\mathbb{Z}$). Then the collection of elements with order p is infinite, and by properties of abelian groups with finite exponent, can be decomposed into infinitely many p-ary direct summands.

7.1.5. Give an example of a first-order theory with e.c. models, where there are e.c. models which satisfy different quantifier-free sentences.

Consider the theory of fields. e.c. models are the algebraically closed fields, as shown in the text. But considering the algebraically closed field of characteristic 2 and the one of characteristic 3, $1+1=0$ is true in the first and not in the second.

7.1.6. Let T be the theory $\forall x \neg(\exists y Rxy \wedge \exists y Ryx)$. Describe the e.c. models of T. Give an example of an $\exists_1$ first-order formula which is not equivalent to a quantifier-free formula in e.c. models of T.

Define $\sigma(x, y) = Rxy \vee Ryx$. Then σ defines a symmetric irreflexive relation on models of T, so graphs can be interpreted in models of T. Thus, if A is an e.c. model of T, then A is the random graph with σ as its connecting relation. In fact, A is e.c. if and only if, for any U, V finite disjoint subsets of T, there is x such that, if $\forall u \in U \forall y(\neg Ruy)$, then $\forall u \in U(Rxu) \wedge \forall v \in V(\neg Rxv)$ and if $\forall u \in U \forall y(\neg Ryu)$, then $\forall u \in U(Rux) \wedge \forall v \in V(\neg Rxv)$. Both directions are obvious. The $\exists_1$ formula is $\exists y(Rxy)$. It is clearly not always true, but there are no non-trivial quantifier-free formulas in x.

7.1.7. In Theorem 7.1.3, suppose that we weaken the assumption of AP to amalgamation over non-empty structures. Show that the conclusion still holds, provided that we require ϕ to have at least one free variable.

Note that the amalgamation property is used in Theorem 7.1.3 given $(A, \bar{a})$ and $(B, \bar{b})$, with an isomorphism taking $\bar{a}$ to $\bar{b}$. Thus, A and B are amalgamated over $\bar{a}$ (with $\bar{a}$ embedding in B as $\bar{b}$). If $\bar{a}$ is not empty, then amalgamation over non-empty structures is sufficient. But $\bar{a}$ is such that $A \models \phi(\bar{a})$. Since ϕ has at least one free variable, $\bar{a}$ must be non-empty.

7.1.8. Let T be complete first-order arithmetic, i.e. the first-order theory of the natural numbers with symbols $0, 1, +, \cdot$. Show that the natural numbers form an e.c. model of T.

Let $\varphi(x, \bar{a})$ be some quantifier-free formula of $\mathbb{N}$. Each element in $\bar{a}$ is definable, so we can rewrite $\varphi(x, \bar{a})$ as $\varphi'(x)$. Then if $T \vdash \exists x \varphi'(x)$, then a witness exists in $\mathbb{N}$. If $T \vdash \neg\exists x \varphi'(x)$, then no witness can exist in any model of T.

7.1.9**. Didn't see in text. Will do shortly (have done in past).

7.2.1. If **K** is an inductive class of L-structure, and **J** is the class of e.c. structures in **K**, show that **J** is closed under unions of chains.

Since **K** is inductive, the union of a chain in **J** is certainly in **K**. We must show then that it is e.c. But any formula $\phi(x, \bar{a})$ has all of its finitely many parameters coming at some finite stage in the chain, and thus there is a witness at that stage, and thus in the union.

7.2.2. A near-linear space is a structure with two kinds of elements, 'points' and 'lines,' and a symmetric binary relation of 'incidence' relating points and lines in such a way that (i) any line is incident with at least two points, and (ii) any two distinct points are incident with at most one line. The near-linear space is a projective plane if (ii') any two distinct points are incident with exactly one line, and moreover (iii) any two distinct lines are incident with at least one point, and (iv) there is a set of four points, no three of which are all incident with a line. (a) Show that the class of near-linear spaces is inductive. (b) Show that every e.c. near-linear space is a projective plane. (c) Deduce that every near-linear space can be embedded in a projective plane.

(a) Let $B = \bigcup_{i<\omega} A_i$ be a union of a chain. Clearly (i) holds. Suppose we have points p, q incident with l, m. But then all elements were added at some finite stage, i, but then A_i is not a near-linear space.

(b) Clearly (ii') and (iii) are necessary conditions for an e.c. near-linear space. Let A be an e.c. near-linear space. Given points p, q, choose a line l with $p \in l$, $q \in l$. Then consider $\exists z(z \notin l)$. It is easily seen to be consistent with $\text{diag}(A)$, so we have r witnessing it. Then let m be the line such that $p \in m, r \in m$, and n be the line such that $q \in n, r \in n$. Then $\exists z(z \notin l \wedge z \notin m \wedge z \notin n)$ gives a final element, s, making p, q, r, s fulfill (iv).

(c) By Theorem 7.2.1, every near-linear space can be embedded in a projective plane.

7.2.3. Show that in Theorem 7.2.4(b), we can replace '$\exists_1$' by 'primitive' (both times).

In the proof of (a) implies (b), a quantifier-free formula $\theta(\bar{c}, \bar{d})$ is produced, from the diagram of A. Write θ as a disjunction of conjunctions. But then one of those conjunctions is true, so we can replace θ by θ' which has only conjunctions of literals. Then $\exists\bar{y}\theta$ is primitive. In the proof of (b) implies (c), with ϕ any $\exists_1$ formula with $C \models \phi(\bar{a})$ for some $C \supseteq A$, if we write ϕ as a disjunction of conjunctions, one of the conjunctions is satisfied by $\bar{a}$, so we can assume ϕ is primitive. Then the rest of the proof goes through.

7.2.4. Let L be a first-order language and T an $\forall_2$ theory in L. (a) Show that if B is a model of T and $A \preccurlyeq_1 B$, then A is a model of T. (b) Show that if B is an e.c. model of T and $A \preccurlyeq_1 B$, then A is an e.c. model of T.

(a) Let $\varphi = \forall\bar{x}\exists\bar{y}\theta(\bar{x}, \bar{y})$ be an axiom of T. For any $\bar{a}$ in A, consider $\exists\bar{y}\theta(\bar{a}, \bar{y})$. Since this is true in B, it is true in A. Thus, φ is true in A.

(b) Suppose $\varphi(\bar{a})$ is $\exists_1$ and true in $C \supseteq A$. By amalgamation, since $(B, A) \Rrightarrow_1 (A, A)$, there is an elementary extension of C, C', with B embedding into C', and the embedding fixing A. Then, since B is e.c., $B \models \varphi(\bar{a})$, so $A \models \varphi(\bar{a})$.

7.2.5. Let L be a first-order language, T an $\forall_2$ theory in L, and A an L-structure. Show that the following are equivalent. (a) A is an e.c. model of T. (b) A is a model of T and for every model C of T such that $A \subseteq C$, we have $A \preccurlyeq_1 C$. (c) For some e.c. model B of T, $A \preccurlyeq_1 B$.

(c) implies (a) by the previous problem, (b) is the definition of (a), and (a) implies (c) trivially, with $A = B$.

7.2.6. Let L be a countable first-order language and T an $\forall_2$ theory in L. Show that there is a sentence ϕ of $L_{\omega_1\omega}$ such that the models of ϕ are exactly the e.c. models of T.

By Theorem 7.2.6, a model A is existentially closed if and only if for every $\exists_1$ formula $\phi(\bar{x})$ $A \models \forall\bar{x}(\phi(\bar{x}) \Leftrightarrow \bigwedge \mathrm{Res}_\phi(\bar{x}))$. This is a sentence of $L_{\omega_1\omega}$. Conjuncting all such sentences, for every ϕ, keeps it $L_{\omega_1\omega}$.

7.2.7. Let L be a first-order language and T an $\forall_2$ theory in L. (a) Show that if A is an e.c. model of T and B is a model of T with $A \subseteq B$, then for every $\forall_2$ formula $\phi(\bar{x})$ of L and every tuple $\bar{a}$ in A, if $B \models \phi(\bar{a})$ then $A \models \phi(\bar{a})$. (b) Show that if B in (a) is also an e.c. model of T then the same holds with $\forall_3$ in place of $\forall_2$.

(a) Suppose $B \models \phi(\bar{a})$, and write $\phi(\bar{x}) = \forall\bar{y}\psi(\bar{x}, \bar{y})$, with ψ $\exists_1$. Then for every choice of $\bar{b}$ in A, $B \models \psi(\bar{a}, \bar{b})$. Since A is e.c., $A \models \psi(\bar{a}, \bar{b})$. Thus, $A \models \phi(\bar{a})$.

(b) Suppose $B \models \phi(\bar{a})$, and write $\phi(\bar{x}) = \forall\bar{y}\exists\bar{z}\psi(\bar{x}, \bar{y}, \bar{z})$, with ψ $\forall_1$. For any $\bar{b}$ in A, then $B \models \exists\bar{z}\psi(\bar{a}, \bar{b}, \bar{z})$. Let $\bar{c}$ be a witness to this. Since $(B, A) \Rrightarrow_1 (A, A)$, we can embed B into an elementary extension of A, A'. Then $B \models \psi(\bar{a}, \bar{b}, \bar{c})$. Since $B \subseteq A'$, $A' \models \psi(\bar{a}, \bar{b}, \bar{c})$, as B is an e.c. model of T. Then $A' \models \exists\bar{z}\psi(\bar{a}, \bar{b}, \bar{z})$, so A satisfies the same thing, so $A \models \phi(\bar{a})$.

7.2.8. Let L be a first-order language and U a $\forall_2$ theory in L. Show that among the $\forall_2$ theories T in L such that $T_\forall = U_\forall$, there is a unique maximal one under the ordering $\subseteq$. Writing U_0 for this maximal T, show that U_0 is the set of those $\forall_2$ sentences of L which are true in every e.c. model of U.

Let T be any model with $T_\forall = U_\forall$. Since every e.c. model of T is an e.c. model of $T_\forall$, and thus an e.c. model of $U_\forall$, and thus an e.c. model of U, every e.c. model of T is an e.c. model of U. Therefore, every sentence in T must be true in every e.c. model of U. Let T be the set of all $\forall_2$ sentences true in every e.c. model of U. If we can show that T satisfies $\forall_T = \forall_U$, then we are done, since T is clearly maximal. Let $\varphi = \forall\bar{x}\theta(\bar{x})$, with θ quantifier-free, be a $\forall_1$ sentence in $T_\forall$. Let A be an e.c. model of U. Consider $\mathrm{diag}(A) \cup \{\neg\varphi\} \cup U$. If it is consistent, then let B be a model. Then $A \subset B$, and $B \models \exists\bar{x}\neg\theta(\bar{x})$, but A does not, but A is e.c. Thus, it cannot be consistent. If $U \cup \{\neg\varphi\}$ is inconsistent, then $U \vdash \varphi$, so $\varphi \in U_\forall$. Otherwise, let $U' = U \cup \{\neg\varphi\}$. Let B be a model of U', and let C be an e.c.

model containing B. Then, since $\neg\varphi$ is $\exists_1$, $C \models \neg\varphi$, which is impossible, since every sentence in $T_\forall$ is true in all e.c. models of U.

7.2.9. Let L be a first-order language (not necessarily countable) and T an $\forall_2$ theory in L. (a) Show that if every model of T is embeddable in a simple model, then every e.c. model of T is simple. (b) Deduce that every e.c. group is simple.

(a) Let A be an e.c. model, and suppose $h : A \to B$ is a homomorphism contradicting the simplicity of A. By assumption, A extends to a simple model C. Let $\bar{a}$ be the elements of A. Then I claim $(C, \bar{a}) \Rrightarrow_1^+ (B, h(\bar{a}))$. $(C, \bar{a}) \Rrightarrow_1 (A, \bar{a})$, since A is e.c., and $(A, \bar{a}) \Rrightarrow_1^+ (B, \bar{a})$, since $A \subseteq B$. Thus, there is an elementary extension D of B and a homomorphism $g : C \to D$ extending h, by Theorem 5.4.7. But since h is not an embedding, g is not an embedding, contradicting the simplicity of C.

(b) Since every group can be embedded into a simple group, every e.c. group is simple by (a).

7.2.10. In commutative rings, write (*) for the condition 'For each positive integer n, there is an element a such that $a^n = 0$ but $a^i \neq 0$ for all $i < n$.' (a) Show that if A is a commutative ring in which (*) holds, and $\phi(x)$ is any formula (maybe with parameters from A) such that $\phi(A)$ is the set of nilpotent elements in A, then A has an elementary extension in which some non-nilpotent element satisfies ϕ. (b) By iterating (a) to form an elementary chain, show that there is a commutative ring in which the set of nilpotent elements is not first-order definable with parameters. (c) Show that every e.c. commutative ring satisfies (*).

(a) By compactness. The set of formulas $\text{eldiag}(A) \cup \{\phi(x)\} \cup \{x^i \neq 0 \mid i < \omega\}$ is finitely consistent, thus consistent, so there is an elementary extension.

(b) L is countable, so if we start with A countable, and start enumerating formulas and taking an elementary extension for each one, we will be done after ω steps, and the union of the extensions will be a commutative ring with no first-order definition of the nilpotent elements, as desired.

(c) Suppose A is a commutative ring. To show that there is some superstructure of A with a nilpotent of degree i, form a new structure of $i \times i$ matrices with coefficients in A. A embeds as the multiples of the identity matrix. Elementary linear algebra shows that we can find a nilpotent element of degree i. Thus, any e.c. commutative ring must have nilpotents of every degree.

7.2.11. Let A be a commutative ring, a an element of A and n a positive integer. Show that the following are equivalent. (a) For every element b of A, if $ba^{n+1} = 0$ then $ba^n = 0$. (b) There is a commutative ring B which extends A and contains an element b such that $(a^2b - a)^n = 0$.

(b) implies (a): Suppose $(a^2b - a)^n = 0$. Choose any c with $ca^{n+1} = 0$. Consider $c(a^2b - a)^n = 0$. Expanding out $(ab - 1)^n$ yields terms where a has degree $\geq n + 1$, except for the last term, $(-a)^n$. Thus, $c(a^2b - a)^n = c(-a)^n$, so we have $-ca^n = 0$ or $ca^n = 0$, so (a) is true.

(a) implies (b): Consider the ring $A[x]/I$, where I is the ideal $(a^2x - a)^n$. We show that this ring contains no non-zero elements of A. Suppose that n is even (the case n is odd is handled similarly).

Let $b = \left(\sum_{i \le m} c_i x^i\right)(a^2x - a)^n$. We show $b = 0$. We can assume $m \ge n$. Multiplying out, we have

$$(c_0 a^n - b) + \sum_{i=1}^{m} x^i \sum_{j=\max(i-n,0)}^{i} \binom{n}{i-j} a^{n+i-j} c_j + \sum_{i=1}^{n} x^{m+i} \sum_{j=i}^{n} \binom{n}{j-1} c_{m-n+j} a^{2n-j+1} = 0$$

So $c_m a^{2n} = 0$. But then, since $c_m a^{n-1} a^{n+1} = 0$, $c_m a^{n-1} a^n = 0$. We can repeat this until we have $c_m a^n = 0$. Thus, $c_{m-1} a^{2n} = 0$, so $c_{m-1} a^n = 0$, and so on down to $c_{m-n+1} a^n = 0$. This continues to propagate down until we have $c_0 a^n = 0$, and so $b = 0$.

7.2.12. Show that the amalgamation property fails for the class of commutative rings without nonzero nilpotents.

Let $A = \mathbb{Q}[x]$. Let B be the field of fractions of A. Let $C = A[y]/(xy)$. Clearly A and B have no nilpotents. C does not either because the radical of (xy) in $\mathbb{Q}[x, y]$ is itself, which is easy to see. Now assume B and C can be amalgamated over A, say in D. Then we have $x \cdot x^{-1} = 1$ in D, and $yx = 0$. But then $y = 0$, which is not true in C, and so cannot be true in D.

7.2.13. Let T ba an $\forall_2$ theory in a first-order language L. Suppose $\phi(\bar{x})$ is an $\exists_1$ formula of L, and $\chi(\bar{x})$ is a quantifier-free formula of $L_{\infty\omega}$ such that for every e.c. model A of T, $A \models \forall \bar{x}(\phi \leftrightarrow \chi)$. Show that χ is equivalent to Res_ϕ modulo T, and hence that Res_ϕ is equivalent modulo T to a set of quantifier-free formulas of L.

We show any model of T which is a model of χ is a model of Res_ϕ and vice versa. Let $(A, \bar{a}) \models T$ be a model of $\chi(\bar{a})$. Then there is an e.c. model of T containing A, B, in which $B \models \chi(\bar{a}) \leftrightarrow \phi(\bar{a}) \leftrightarrow \mathrm{Res}_\phi(\bar{a})$. Then $B \models \mathrm{Res}_\phi(\bar{a})$. Since Res_ϕ consists of $\forall_1$ formulas, $A \models \mathrm{Res}_\phi(\bar{a})$. The reverse argument is similar.

Now, this means that all of the formulas in Res_ϕ are satisfied by $\bar{a}$ if and only if $\chi(\bar{a})$, in models of T. Write χ as a disjunction of conjunctions. By compactness, for any one formula of Res_ϕ, ψ, ψ is equivalent to a formula formed by taking only a finite number of disjunctions in χ, and then a finite number of conjunctions in those disjunctions – a first-order quantifier-free formula. Thus, every formula in Res_ϕ is equivalent to a quantifier-free formula.

7.3.1. Show that if L is a first-order language, T and U are theories in L, $T \subseteq U$ and T is model-complete, then U is also model-complete.

Let $A \subseteq B$ be models of U, hence models of T. Then $A \preccurlyeq B$. Since this is true for any two such models of U, U is model-complete.

7.3.2. Show that if a first-order theory T is model-complete and has the joint-embedding property, T is complete.

Let A and B be any two models of T. Let C be a model in which both A and B embed. Then $A \preccurlyeq C$ and $B \preccurlyeq C$, so $A \equiv C \equiv B$.

7.3.3. In the first-order language whose signature consists of one 1-ary function symbol F, let T be the theory which consists of the sentences $\forall x F^n(x) \neq x$ (for all positive integers n) and $\forall x \exists_{=1} y F(y) = x$. Apply Lindström's test to show that T is model-complete.

$\forall x \exists_{=1} y F(y) = x$ can be written as $\forall x y_1 y_2 (F(y_1) = x \wedge F(y_2) = x \to y_1 = y_2)$. Thus, T is $\forall_2$. The first axiom insures that T has no finite models. T says that F defines $\mathbb{Z}$-chains. Thus, a model of T is just a union of $\mathbb{Z}$-chains. Any 2 models of cardinality ω_1 are then isomorphic, since they consist of ω_1 copies of $\mathbb{Z}$. Thus, T is model-complete.

7.3.4*. Let T be the theory of (non-empty) linear orderings in which each element has an immediate predecessor and an immediate successor, in a language with relation symbols $<$ for the ordering and $S(x, y)$ for the relation 'y is the immediate successor of x.' Show that T can be written as an $\forall_2$ theory. Show by Lindström's test that T is model-complete. Deduce that T is complete.

We can axiomatize T by the axioms for a linear order, which are $\forall_1$, along with: $\forall x(\exists_{=1} y S(x, y) \wedge \exists_{=1} y S(y, x))$ and $\forall xyz((S(x, y) \to (z > x \to z \geq y))) \wedge (S(y, x) \to (z < x \to z \leq y))$. These axioms are $\forall_2$. T is not complete by Lindström's test, since for any cardinal κ, $\kappa \times \mathbb{Z}$ and $\kappa^* \times \mathbb{Z}$ are non-isomorphic. However, any superstructure of a discrete linear ordering must just add more $\mathbb{Z}$-chains, and it is easy to see that such an embedding is elementary, by a back-and-forth argument, so T is model-complete. Since it is easy to jointly embed two models, A and B into some model C (let $C = A + B$, and put all elements of B greater than those of A), T is complete.

7.3.5. Give an example of a theory T in a first-order language L, such that T is not model-complete but every complete theory in L containing T is model-complete.

Let T be the theory of a unary function, which is either like that in Exercise 7.3.3, or maps every element to itself. T is the theory $\forall x(F(x) = x) \vee \forall x \exists_{=1} y(F(y) = x)$, along with $\forall x(F(x) = x) \vee \forall x(F^n(x) \neq x)$, for each n. Then each completion of T is clearly model-complete and complete, but they are also clearly different, so T is not complete.

7.3.6. Suppose T is a theory in a first-order language, and every completion of T is equivalent to a theory of the form $T \cup U$ for some set U of $\exists_1$ sentences. Suppose also that every completion of T is model-complete. Show that T is model-complete.

Let $A \subseteq B$ be models of T. $\mathrm{Th}(A)$ is a completion of T, and hence equivalent to $T \cup U$ for U a set of $\exists_1$ sentences. Then, since $\exists_1$ sentences pass upwards, B is also a model of $T \cup U$, hence $A \preccurlyeq B$.

7.3.7. Give an example of a theory T in a countable first-order language, such that T has 2^ω pairwise non-isomorphic algebraically prime models.

Let L be a language with 2 unary function symbols and countably many relation symbols, $\{f, g\} \cup \{P_i \mid i < \omega\}$. Let T be the theory which says: 'Every element satisfies at most one P_i;' 'Every P_i is nonempty;' For each i, and for any term t, $\forall x(P_i(x) \to P_i(t(x))$; For each i, and for any terms s and t, $\forall xy(x \neq y \to t(x) \neq s(y))$.

Thus, in each P_i, we have a term algebra. For each $k < \omega$, let A_k be the model in which, for $i \neq k$, P_i is a term algebra over one variable, x_i, and P_k is a term algebra over two variables, x_k and y_k. I claim each A_k can be embedded in any $B \models T$. For each $i < \omega$, find $b \in B$ with $B \models P_i(b)$. If $i \neq k$, map x_i to b. This induces an embedding of $P_i(A_k)$ into $P_i(B)$. For $i = k$, map x_k to $f(b)$ and y_k to $g(b)$. Then this induces an embedding of $P_k(A_k)$ into $P_k(B)$. Thus, A_k embeds into B, and so is algebraically prime. However, $A_i \not\cong A_j$, since if $x_i \in A_i$ maps to $x_i \in A_j$, then y_i must become a term of x_i, which it wasn't before, and vice versa. But then nothing can map to $x_i \in A_j$, and so there is no isomorphism.

7.3.8. Let T be the theory which says the following. All elements satisfy exactly one of $P(x)$ and $Q(x)$; for every element a satisfying $P(x)$ there are unique elements b and c such that $R(b, a)$ and $R(a, c)$; $R(x, y)$ implies $P(x)$ and $P(y)$; there are no finite R-cycles; $S(x, y)$ implies $P(x)$ and $Q(y)$; if $R(x, y)$ and $Q(z)$ then $S(x, z)$ iff $S(y, z)$. Show that in an e.c. model of T, elements a, b satisfying $P(x)$ are connected by R if and only if there is no element c such that $S(a, c) \leftrightarrow \neg S(b, c)$. Deduce that T has e.c. models with elementary extensions which are not e.c., and hence that T is not companionable.

Models of T consist of $\mathbb{Z}$-chains, along with elements which are associated with some chain(s) (possibly none). In a model of T, A, we know that if a and b are connected by R, then $S(a, c) \leftrightarrow S(b, c)$ for all $c \in Q(A)$. However, if a and b are not connected, it might still be that $S(a, c) \leftrightarrow S(b, c)$ for all $c \in Q(A)$. However, it is trivial to embed A in a model which has some c as a counterexample, and this means that A is not e.c. Thus, any e.c. model of T must have such a counterexample for any a and b not connected, so the claim holds. However, if A is e.c., we can just adjoin a new $\mathbb{Z}$-chain and set $S(a, x) = S(b, x)$ for every a in the new chain, and b in some old chain. It is easy to see that this is an elementary extension, since we can play a back-and-forth game, but it is not e.c. since it fails the above property. Thus, the class of e.c. structures is not closed under elementary equivalence. Since T is an $\forall_2$ theory, this means that T is not companionable.

7.3.9. Let G be a group and a, b two elements of G. (a) Show that the following are equivalent. (i) There is a group $H \supseteq G$ with an element h such that $h^{-1}ah = b$. (ii) The elements a and b have the same order. (b) Explain this as an instance of Lemma 7.2.5. (c) Prove that the theory of groups is not companionable.

(a) Clearly (i) implies (ii), since then $b^n = h^{-1}a^n h$, which is 1 iff $a^n = 1$. For the converse, consider G acting on itself by left multiplication. G can then be embedded onto a subgroup of the symmetric group on $|G|$ elements, denoted Ω. Any element $a \in G$ can be regarded as a (possibly infinite) product of disjoint (possibly infinite) cycles. All of these cycles have the same order. To see this, let g and h be elements in different cycles. WLOG, assume g's cycle is finite (if there are no finite cycles, we are done). Then for some n, $a^n g = g$, so $a^n h = a^n(gg^{-1}h) = (a^n g)g^{-1}h = h$. Thus h's cycle has length at most n. Repeating the argument with h and g reversed shows that their cycles have the same length. Then if two elements of G, a and b, have the same order, their cycles clearly all have the same length. With that, it is trivial to construct an element of Ω, h, such that $h^{-1}ah = b$.

(b) T, the theory of groups, is an $\forall_2$ theory, and the formula we wish to satisfy is $\exists_1$. Thus, we can expand G to such an H iff the resultant is true. The resultant is exactly those formulas which say that a and b have the same order.

(c) If T had a companion, T^*, we must have that $\mathrm{Res}_\phi(x, y)$ is equivalent modulo T^* to a single $\forall_1$ formula $\psi(x, y)$ of L, by Corollary 7.3.7. Then $\psi(x, y)$ must say that x and y have the same order. $T^* \cup \{\neg\psi(x, y)\} \cup \{x^n \neq 1 \wedge y^n \neq 1 \mid 0 < n < \omega\}$ cannot be consistent, since x and y have the same infinite order. Since there certainly are e.c. groups with elements of infinite order, we have $T^* \vdash \bigwedge_{i<m} x^{n_i} \neq 1 \wedge y^{n_i} \neq 1 \rightarrow \psi(x, y)$ by compactness, for some finite m. But then, taking x and y with orders not equal, but both larger than $\max(n_i \mid i < m)$, we have $\psi(x, y)$, which is impossible.

7.3.10. Give an example of a ω_1-categorical countable first-order theory T in a countable first-order language L, such that no definitional expansion of T by adding finitely many symbols is model-complete.

Let L be the language with countably many equivalence relations, $\langle E_i \mid i < \omega\rangle$. Let A be the L-structure with universe ω, with the classes of E_i being $\{0\}, \{1, 2\}, \{3, 4, 5, 6\}, \ldots \{2^i - 1, \ldots, 2^{i+1} - 2\}$ along with $\{2^{i+1} - 1 + k(2^{i+1}), \ldots, 2^{i+1} - 1 + (k + 1)2^{i+1} - 1\}$ $(k < \omega)$. Thus, for each E_i, there is a class with 1 element, a class with 2, and so on up to a class with 2^i elements. Thereafter, every class has 2^{i+1} elements. If an element is in a "distinguished" E_i class (in a class with $< 2^{i+1}$ elements), then it is in a distinguished E_j class for every $j > i$, but the converse has counterexamples. As well, if xE_iy, then xE_jy for $j > i$. A is clearly the prime model for T. We define an L-structure C. C looks like A, but has no distinguished classes. Thus, for each E_i, every E_i equivalence class of C has 2^{i+1} elements. xE_iy still implies xE_jy for $j > i$. If B is an uncountable model of T, it is easy to see that B breaks up into ω along with copies of C. Two elements are in the same copy of C iff there is some E_i such that they are in the same equivalence class of E_i. Since each E_i has finitely many elements in each equivalence class, C is countable, so B must have ω_1 copies of C. Thus, T is ω_1-categorical. However, T is not model-complete for any finite definitional expansion of L. Let B be any model of T. Let $\{\varphi_1, \ldots, \varphi_n\}$ be the formulas used in a definitional expansion (which we can assume is relational), and let k be the greatest index of an equivalence relation appearing in the φ's. Now take a such that xE_ka has 2^{k+1} elements satisfying it and so does $xE_{k+1}a$ (a is in a distinguished class for E_{k+1} but not for E_k). Let $D = B + C$ be the structure with B and also with a copy of C, the structure defined above. Now we consider a mapping of B into D, g. All elements are mapped to themselves, except for the E_k equivalence class of a, which is mapped to an arbitrary E_k class of C. Now, consider $\varphi_i(\bar{b}, \bar{a})$, for some $\bar{a}$ in the equivalence class of a, and $\bar{b} \in B$. A back-and-forth argument shows that $B \models \varphi_i(\bar{b}, \bar{a}) \Leftrightarrow D \models \varphi_i(\bar{b}, g(\bar{a}))$. Thus, g preserves the new definitions. g is clearly an embedding of the original language, so g is an embedding. But $g(B) \not\preccurlyeq D$, since a no longer has exactly 2^{k+1} elements in its E_{k+1} equivalence class.

7.3.11. Give an example of an ω_1-categorical countable first-order theory which is not companionable.

Consider the model, A, of ordered pairs, (m, n), $m \geq -1$, $n \geq 0$. There is a unary relation symbol, $Diag$, picking out the elements (m, m). There are two symmetric binary relation symbols, $Horiz$ and $Vert$. The relation $Horiz$ is an equivalence relation; its classes are the sets $\{(m, b) \mid m \geq -1\}$ with b fixed. The relation $Vert$ holds between (m, n) and (m, m) whenever $m, n \geq 0$. Let $T = \mathrm{Th}(A, \bar{a})$, where $\bar{a}$ lists the elements of A. I claim T is ω_1-categorical and not companionable. The sentence $\forall x_1, x_2(\forall y(\neg Vert(x_1, y) \wedge \neg Vert(x_2, y)) \rightarrow \neg Horiz(x_1, x_2))$, along with $\forall x \exists y(Diag(y) \wedge Horiz(x, y))$ and some other obvious sentences, assures us that $Diag$ has cardinality ω_1 in a model of size ω_1. Then every set has size ω_1, and it is easy to see that all such models are isomorphic. However, let U be a putative model companion for T. Then U says that infinitely many named elements are not paired with other elements by $Vert$, since $U_\forall = T_\forall$, so every model of U has an elementary extension with new such elements. Let B be a model of U and B' the extension. Take $C \supseteq B'$ a model of T, and then take $D \supseteq C$ a model of T such that these new elements have $Vert$ pairings. Then take $E \supseteq D$ a model of U, showing that U is not e.c., and so not a companion.

7.3.12. Give an example of a companionable $\forall_1$ theory T in a first-order language L, and an $\exists_1$ formula $\phi(\bar{x})$ in L such that Res_ϕ is not equivalent modulo T to any finite set of $\forall_1$ formulas.

Let T be the set of sentences $\forall xy \neg(P_0 x \wedge P_i y)$, $(0 < i < \omega)$, where the P_i's are unary relations, and let ϕ be $\exists x P_0 x$. Then clearly, a model A has an expansion satisfying ϕ iff for every $i > 0$, $A \models \forall x \neg P_i x$. By compactness, there is no way to express this as a first-order $\forall_1$ sentence. We now show T is companionable – its e.c. models come in two kinds. Either there are infinitely many elements which do not satisfy P_0 and infinitely many which do satisfy P_0 or there are infinitely many elements which satisfy any boolean combinations of the P_i's, $i > 0$. These axioms can be written as $\exists_{\geq n} x \neg P_0 x$ and $\exists_{\geq n} x(P_0(x) \vee \bigwedge_{0<i<n} P_i^{\sigma(i)} x)$, where σ is a map from n to 2, along with the axioms of T. Thus, the e.c. models are axiomatizable, and thus T is companionable.

7.4.1. In Theorem 7.4.1, show that T has quantifier elimination if and only if condition (c) holds whenever $\bar{a}$ is a tuple of elements of A.

Theorem 7.4.1 proves that if, when A and B are models of T, $\bar{a}$ a sequence from A, and $e : \langle \bar{a} \rangle_A \rightarrow B$ is an embedding, then there is always an elementary extension D of B and an embedding $f : A \rightarrow D$ which extends e, then T has quantifier elimination, and vice versa. Clearly, if $\bar{a}$ is a tuple, then T implies the modified (c) (the if-clause), since a tuple is still a sequence. The question is whether the modified (c) implies quantifier elimination. Thus, we know that we can extend every tuple to a full embedding, but perhaps not every sequence. However, we show that given $\varphi(\bar{x})$, any formula, the truth of $\varphi(\bar{a})$ depends only on the quantifier-free type of $\bar{a}$. Go by induction on the number of quantifiers in φ. Let A be any model, with $\bar{a}$ in A. Suppose $A \models \varphi(\bar{a})$. Let B be a model with $e(\langle\bar{a}\rangle_A)$ an embedding of $\langle\bar{a}\rangle_A$ in B. Then $\bar{b} = e(\bar{a})$ satisfies the same quantifier-free type. We can assume that e is the identity, so $\bar{b} = \bar{a}$. Now, by (c), take B an elementary extension of B and f an extension of e, so that $f(A) \subseteq D$. If $\varphi(\bar{x}) = \exists y\theta(\bar{x}, y)$, where θ has 1 fewer quantifier, then choose $c \in f(A)$ satisfying $f(A) \models \theta(\bar{a}, c)$. Then by induction, since $\langle\bar{a}\rangle_{f(A)}$ embeds into D, $D \models \theta(\bar{a}, c)$. Thus, $D \models \varphi(\bar{a})$, so $B \models \varphi(\bar{a})$. If the first quantifier of φ is $\forall$, then since we have an embedding of $\langle\bar{a}\rangle_D$ into A (the identity), we can elementarily extend A to C containing a copy of D (Note that the copy of A in the copy of D may not coincide with the A in C, but the $\bar{a}$ will). Then since $C \models \varphi(\bar{a})$, $C \models \theta(\bar{a}, c)$ for each c in the copy of D, if $\varphi(\bar{x}) = \forall y\theta(\bar{x}, y)$. Thus, $D \models \varphi(\bar{a})$, and so $B \models \varphi(\bar{a})$. Thus, $\varphi(\bar{a})$ is determined by the quantifier-free type of $\bar{a}$. By an application of compactness, φ is thus equivalent to a single quantifier-free formula.

7.4.2*. Show that in Corollary 7.4.3, condition (a) can be replaced by (a'): if B is a model of T and A is a proper substructure of B, then there are an element b of B which is not in A and a set $\Phi(x)$ of quantifier-free formulas of L with parameters in A, such that $B \models \bigwedge \Phi(b)$, Φ determines the quantifier-free type of b over A, and for every finite subset Φ_0 of Φ, $A \models \exists x \bigwedge \Phi_0$.

This condition is not fulfilled by real closed fields. Anyway, following the proof of Corollary 7.4.3, all we need to show is that given a model, B, and a submodel, C, any element in B induces a consistent quantifier-free type in C. Let b be any element of $B \setminus C$. First, define $A \supseteq C$ to be a maximal substructure of A not containing b. Such a A exists just by repeatedly adjoining elements of B, and stopping when no more can be adjoined without including b. Now, note that the quantifier-free type of b is consistent with A: since any element not in A cannot be added, any element $d \in B \setminus A$ must, with some $\bar{a}$ in A, define b with some term, $t(x, \bar{y})$ – otherwise, we could add d. Fix d with a consistent quantifier-free type over A (exists by the statement of the problem). Then, if $\Phi(x)$ is the complete quantifier-free type of b over A, we know that $A \models \exists x \bigwedge \Phi_0(t(x, \bar{a}))$ for Φ_0 any finite subset of Φ. Thus, $\exists y \bigwedge \Phi_0(y)$, just setting y to be $t\left(b'_{\Phi_0}, \bar{a}\right)$, where b'_{Φ_0} is the witness. Now, suppose the quantifier-free type of b over C is not consistent with C. Then for some $\phi(x, \bar{c})$, a finite quantifier-free formula, there is no witness in C. But there is certainly a witness in A, say e. Consider $\langle C \cup \{e\}\rangle_A$. Then by the same argument as above, the quantifier-free type of e is consistent over C. But the quantifier-free type of e says $\phi(x, \bar{c})$. Thus, there must be a witness in C. The rest of the Corollary proceeds much the same way: since the quantifier-free type of b is consistent with C, it is consistent with any supermodel of C, so some elementary extension of any supermodel contains a witness.

7.4.3. Let L be a first-order language with finite signature, and T a theory in L. Show that the following are equivalent. (a) T has quantifier elimination. (b) If A and B are any models of T, then for each $n < \omega$, any pair of tuples $(\bar{a}, \bar{b})$ from A, B, respectively, such that $(A, \bar{a}) \equiv_0 (B, \bar{b})$, is a winning position for player $\exists$ in the game $G_n[A,B]$.

It is easy to show that $(\bar{a},\bar{b})$ is a winning position for $G_n[A,B]$ iff they satisfy all the same unnested formulas of quantifier rank n. Then (a) easily implies (b), $\bar{a}$ and $\bar{b}$ actually agree on every formula. Moreover, if $\varphi(\bar{x})$ is any formula, φ is equivalent to θ, with θ unnested and of finite quantifier rank, say n. Since $(\bar{a},\bar{b})$ is a winning position for $G_n[A,B]$, θ agrees on $\bar{a}$ and $\bar{b}$, so φ does as well. Thus, the quantifier-free type determines φ. An application of compactness later, φ is quantifier-free.

7.4.4. Let L be a first-order language and T a theory in L. Show that the following are equivalent. (a) T has quantifier elimination. (b) if A and B are any ω-saturated models of T and $\bar{a}$ a tuple of elements of T such that $(A,\bar{a}) \equiv_0 (B,\bar{a})$, then $(A,\bar{a})$ and $(B,\bar{a})$ are back-and-forth equivalent. (c) If A is a model of T, B is a λ-saturated model of T for some infinite cardinal $\lambda > |A|$, and $\bar{a}$ is a tuple of elements of T such that $(A,\bar{a}) \equiv_0 (B,\bar{a})$, then there is an elementary embedding $f: A \to B$ such that $f\bar{a} = \bar{b}$.

(a) implies (b) is just applying the definition of the back-and-forth game, since given $(A,\bar{a}) \equiv_0 (B,\bar{b})$, we know that $(A,\bar{a}) \equiv_1 (B,\bar{b})$, so if c is any element of A, the type of c over $\bar{a}$ is consistent over $\bar{b}$, and so realized in B, so $(A,\bar{a}c) \equiv_0 (B,\bar{b}d)$, for some d. We can continue this, so A and B are back-and-forth equivalent. For the converse, given any $\varphi(\bar{x}) = \forall y_0 \forall y_1 \exists y_2 \ldots \forall y_n \theta(\bar{y},\bar{x})$, if $A \models \varphi(\bar{a})$ and $B \models \neg\varphi(\bar{a})$, we extend A and B to ω-saturated models, then play a back-and-forth game, choosing $\bar{y}$. Let φ_i be the φ with the first i quantifiers deleted. If we have chosen $c_0,\ldots,c_i$, and $d_0,\ldots,d_i$, with $(A,\bar{a},c_0,\ldots,c_i) \equiv_0 (B,\bar{a},d_0,\ldots,d_i)$, $A \models \varphi_i(c_0,\ldots,c_i,\bar{a})$ and $B \models \neg\varphi_i(d_0,\ldots,d_i,\bar{a})$. If φ_i's outer quantifier is "$\exists$," have $\forall$ choose a witness from A to this formula. Then, when $\exists$ has made the proper move from B according to the winning strategy, both of the above properties are preserved, since $B \models \forall y_{i+1} \neg\varphi_{i+1}(d_0,\ldots,d_i,y_{i+1},\bar{a})$, so any choice by $\exists$ for y_{n+1} preserves $\neg\varphi_{i+1}$. After n moves, we will have two quantifier-free formulas, and $\bar{a}\bar{c}$ and $\bar{a}\bar{d}$ will disagree on them, but this is impossible, since we will have $(A,\bar{a}\bar{c}) \equiv_0 (B,\bar{a}\bar{d})$. Thus, the assumption that we could find such $(A,\bar{a})$ and $(B,\bar{a})$ was wrong, so the quantifier-free type does determine φ, and compactness means that φ is quantifier-free.

For (c) implies (a), given $(A,\bar{a})$ and $(B,\bar{a})$, elementarily extend B to a λ-saturated model, B', then elementarily embed A in it. Then A and B' agree on $\varphi(\bar{a})$ for any φ, and B' and B agree, so A and B agree. Thus, quantifier-free type determines satisfaction, so by compactness, φ is quantifier-free. For (a) implies (c), we know that, since $(A,\bar{a}) \Rrightarrow_1 (B,\bar{a})$ (since $(A,\bar{a}) \equiv_1 (B,\bar{a})$), we can embed A in B, sending $\bar{a}$ to $\bar{a}$, by Theorem 8.3.1. The embedding is elementary since T has quantifier elimination.

7.4.5. Let L be a first-order language, T a theory in L and $\phi(\bar{x})$ a formula of L. Show that the following are equivalent. (a) ϕ is equivalent modulo T to a quantifier-free formula $\psi(\bar{x})$. (b) If A and B are any two models of T and $\bar{a}$, $\bar{b}$ are tuples of elements of A, B respectively such that $(A,\bar{a}) \equiv_0 (B,\bar{b})$, then $A \models \phi(\bar{a})$ implies $B \models \phi(\bar{a})$. (c) If A and B are any two models of T, $\bar{a}$ is a tuple of elements of A such that $A \models \phi(\bar{a})$, and $f: \langle\bar{a}\rangle_A \to B$ is an embedding, then $B \models \phi(f\bar{a})$.

(b) and (c) are clearly equivalent, since if there is an embedding, the tuples are 0-equivalent, and if they are 0-equivalent, there is an embedding. (b) implies (a) by compactness. (a) implies (b) trivially.

7.4.6. Let L be a first-order language and T a complete theory in L. Show that T has quantifier elimination if and only if every model of T has an ultrahomogeneous elementary extension.

In the forwards direction, we know that every model of T has a strongly ω-homogeneous elementary extension. For such a model, say M, to be ultrahomogeneous, we need that if $(M, \bar{a}) \equiv_0 (M, \bar{b})$, then in fact $(M, \bar{a}) \equiv (M, \bar{b})$. Quantifier elimination gives this to us, and so M is ultrahomogeneous. In the reverse direction, given any model A, with $\bar{a}$ and $\bar{b}$ tuples in A such that $(A, \bar{a}) \equiv_0 (A, \bar{b})$, take an ultrahomogeneous elementary extension, M. Then $(M, \bar{a}) \equiv_0 (M, \bar{b})$, so there is an automorphism of M taking $\bar{a}$ to $\bar{b}$. Thus, for any formula φ, $M \models \varphi(\bar{a}) \Leftrightarrow M \models \varphi(\bar{b})$, so the same is true of A. Thus, in any model of T, the quantifier-free type of a tuple determines which formulas it satisfies. If two models with tuples of the same quantifier-free type disagree on some formula on those tuples, extend one of the models to a λ-universal model with λ larger than the cardinality of the other model, and then embed the other model in this universal one. Then a single model disagrees on tuples, which is impossible. Thus, the quantifier-free type determines satisfaction of every formula, so modulo T, by compactness, all formulas are quantifier-free.

7.4.7*. Let A be an integral domain. Show that if A is q.e. then A is a field.

If A is finite, then A is trivially a field, by taking powers of each element. So let A be infinite. The formula $\exists y(xy = 1)$ defines the set of invertible elements in A. By quantifier elimination, it is equivalent to a quantifier-free formula, which of necessity is a finite disjunction of conjunctions of formulas saying that polynomials in x are equal or inequal to 0. By embedding A in its field of fractions, K, it is easy to see that limits on the number of roots that we know for fields still hold for integral domains. There is some conjunction in this formula which holds for infinitely many elements of A, by the pigeonhole principle, but if any formula in the conjunction has a non-trivial equality, then only finitely many elements of A can satisfy it. Thus, the conjunction contains only inequalities, so only finitely many elements (the roots of the polynomials concerned) can be non-invertible. But if a is any invertible element, and b is a non-invertible element, then ab must also be non-invertible, since if it had an inverse, $(ab)^{-1}$, then $(ab)(ab)^{-1} = 1$, so $b(a(ab)^{-1}) = 1$, so b is invertible. Thus, for every such b, we can find a_1 and a_2 distinct such that $a_1 b = a_2 b$, by the pigeonhole principle again, so then $(a_1 - a_2)b = 0$. But this violates the fact that A is an integral domain unless $b = 0$. Therefore the only non-invertible element is 0, so A is a field.

7.4.8. Let T be a first-order theory with a model companion U. Show that U has quantifier elimination if and only if $T_\forall$ has the amalgamation property.

By Theorem 7.4.1, since U is model-complete, if $U_\forall = T_\forall$ has the amalgamation property, U

eliminates quantifiers. Conversely, also by Theorem 7.4.1, if U eliminates quantifiers, then U is model complete (already known) and $U_\forall = T_\forall$ has the amalgamation property.

7.4.9. Let T be a theory in a first-order language L, and U a model companion of T. Show that the following are equivalent. (a) T has the amalgamation property. (b) For every model A of T, $U \cup \text{diag}(A)$ is a complete theory in $L(A)$. (A theory U satisfying (a) or (b) is said to be a model-completion of T.)

(a) implies (b): Let A be any model of T. Let B and C be any two models of $U \cup \text{diag}(A)$. Since U is a model companion, we can find models B' and C' of T containing B and C, respectively. Then we can amalgamate B' and C' into D', a model of T, preserving A. Finally, extend D' to D, a model of U. Since U is model-complete, $B_1 \preccurlyeq D$ and $B_2 \preccurlyeq D$. Thus, if $\bar{a}$ is a listing of all elements of A, $(B_1, \bar{a}) \equiv (B_2, \bar{a})$. Thus, $U \cup \text{diag}(A)$ is complete.

(b) implies (a): Let A be any model of T, and let B and C be models of T containing A. Let B' and C' be models of U containing B and C respectively. Since $U \cup \text{diag}(A)$ is complete, and both B' and C' must satisfy it, $(B', \bar{a})$ and $(C', \bar{a})$ are elementarily equivalent, where $\bar{a}$ is a listing of A. Thus, by amalgamation, there is an elementary extension of B', D', and an embedding of C' into D' which preserves $\bar{a}$. Since D' elementarily extends B', it is a model of U. Let D be a model of T containing D'. Then $B \subseteq D$, and C embeds into D by the embedding of C' into D', and A is preserved by the embedding.

7.4.10. Let L be the first-order language whose signature consists of one 1-ary function symbol. Show that the empty theory in L has a model completion.

It is trivial to see that the empty theory has the amalgamation property. Thus, if we can show that the e.c. models are axiomatizable, then we are done. Looking at $\exists_1$ formulas of L, if in some model every element is the image of infinitely many other elements, and there are infinitely many cycles of every finite order, then that model is existentially closed, and vice versa. These properties are axiomatized by $\forall x \exists_{>n} y(f(y) = x)$ and $\exists_{>n} y(f^m(y) = y)$, for all $n, m \in \omega$.

7.4.11*. An ordered abelian group is an abelian group with a 2-ary relation $\leq$ satisfying the laws '$\leq$ is a linear ordering' and $\forall xyz(x \leq y \to x+z \leq y+z)$. Let T_{oa} be the first-order theory of ordered abelian groups and T_{doa} the first-order theory of ordered abelian groups which are divisible as abelian groups. (a) Show that T_{doa} is the model companion of T_{oa}. (b) Show that T_{doa} has quantifier elimination and is complete. (c) Show that $T_{\text{doa}} = \text{Th}(\mathbb{Q}, \leq)$ where $\mathbb{Q}$ is the additive group of rationals and $\leq$ is the usual ordering.

It is easy to see that any ordered abelian group can be made divisible, by the same argument as in Exercise 7.1.4. (In fact, there are no cyclic groups in the decomposition.) Thus, the e.c. models of T_{oa} must be divisible. Thus, if we can show that T_{doa} is model-complete, we will be done. We show this by showing that T_{doa} has elimination of quantifiers. Consider $\exists y \theta(\bar{x}, y)$, with θ quantifier-free. Writing

θ as a disjunction of conjunctions and passing $\exists y$ through the disjunction, we can assume that θ is a conjunction of literals. Now, if any literal with y in it is an equality, then we can write $ny = t(\bar{x})$, with $n \in \mathbb{Z} \setminus \{0\}$, and t some term. Note that $x \to |n|x$ is an embedding. Let θ' be the result of replacing every variable x in θ with $|n|x$, and replacing every $|n|y$ with $t(\bar{x})$ (if n is negative, put $t(\bar{x})$ on the other side of the equality or inequality in question). θ' does not mention y. Now, suppose $A \models \theta'(\bar{a})$, with $A \models T_{\mathrm{doa}}$ and $\bar{a}$ a tuple in A. Then, letting c be the unique element such that $nc = t(\bar{a})$, it is easy to see that $A \models \theta(c, \bar{a})$. Reversing, if c satisfies $\theta(y, \bar{a})$, then we know that $nc = t(\bar{x})$, and the rest follows. Thus, we may assume that no literal in θ is an equality. Using the shorthand $x < y$ for $x \leq y \wedge x \neq y$, we see that by possibly rearranging θ and passing the $\exists y$ quantifier through again, we can assume that all literals are of the form $t(\bar{x}, y) < s(\bar{x}, y)$, for terms s, t. Let there be k literals in θ. We can write every literal involving y in the form $n_i y < t(\bar{x})$ or $t(\bar{x}) < n_i y$, for some $n_i \in \mathbb{N} \setminus \{0\}$, $i < k$. Let $m = \mathrm{lcm}(n_i \mid i < m, n_i \neq 0)$. Replacing each variable in the ith term by m/n_i copies (leaving it unaltered if $n_i = 0$), we have a finite number of order conditions on a single element, with respect to some terms of $\bar{x}$. Since it is easy to see that a divisible ordered group is a dense linear ordering, quantifier elimination for dense linear orders shows that the existence of such an element only depends on the ordering of these terms of $\bar{x}$, which is expressible with a quantifier-free formula. Thus, θ is equivalent to a quantifier-free formula, so T_{doa} eliminates quantifiers. Thus, it is model-complete, and also complete. Since $(\mathbb{Q}, \leq)$ is a divisible ordered abelian group, and T_{doa} is complete, $\mathrm{Th}(\mathbb{Q}, \leq)$ must equal T_{doa}.

7.4.12. In the notation of the previous exercise, show that all ordered abelian group satisfy the same $\exists_1$ first-order sentences.

We show that $(\mathbb{Q}, \leq)$ and $(\mathbb{Z}, \leq)$ satisfy the same $\exists_1$ first-order sentences. Since $(\mathbb{Q}, \leq)$ contains $(ZZ, \leq)$, only one direction needs to be shown. Suppose $\exists \bar{x} \theta(\bar{x})$ is an $\exists_1$ sentence, and let $\bar{a}$ in $\mathbb{Q}$ witness its truth in $\mathbb{Q}$. Multiplying $\bar{a}$ by $\gcd(\bar{a})$ to get $\bar{b}$ (this multiplication is an embedding), we see that $\mathbb{Q} \models \theta(\bar{b})$, and since $\bar{b}$ is in $\mathbb{Z}$, $\mathbb{Z} \models \theta(\bar{b})$, so $\mathbb{Z} \models \exists_1 \bar{x} \theta(\bar{x})$. Thus, $(\mathbb{Z}, \leq)$ and $(\mathbb{Q}, \leq)$ satisfy the same $\exists_1$ sentences. But it is easy to see that any ordered abelian group contains $(\mathbb{Z}, \leq)$ as an ordered abelian group, so any ordered abelian group satisfies all the $\exists_1$ sentences that $(\mathbb{Z}, \leq)$ satisfies, and any ordered abelian group is contained in a divisible ordered abelian group, whose theory is the same as $\mathrm{Th}(\mathbb{Q}, \leq)$, so $(\mathbb{Q}, \leq)$ satisfies all the same $\exists_1$ sentences that the group satisfies, and so every ordered abelian group satisfies exactly the same $\exists_1$ sentences.

7.4.13. Let L be a first-order language and T an $\forall_1$ theory in L. Show that if $\phi(\bar{x})$ is a formula of L such that both ϕ and $\neg\phi$ are preserved by all embeddings between models of T, then ϕ is equivalent modulo T to a quantifier-free formula $\psi(\bar{x})$.

Let A be a model, and $\bar{a}$ a tuple in A with $A \models \phi(\bar{a})$. If such cannot be found, then ϕ is trivially quantifier-free. Now let $B = \langle \bar{a} \rangle_A$. Since T is $\forall_1$, B is a model of T, and since $B \subseteq A$, $B \models \phi(\bar{a})$. But

B embeds into any model C that contains $\bar{c}$ with the same quantifier-free type as $\bar{a}$. Thus $C \models \phi(\bar{c})$. Thus, whether a tuple satisfies ϕ depends only on the quantifier-free type of the tuple. By compactness, ϕ is quantifier-free.

7.4.14. Let L be a first-order language, T an $\forall_1$ theory in L and $\phi(\bar{x})$ a quantifier-free formula of L. Show that if $T \vdash \exists\bar{x}\phi$, then for some $m < \omega$ there are tuples of terms $\bar{t}_0(\bar{y}), \ldots, \bar{t}_{m-1}(\bar{y})$ such that $T \vdash \forall\bar{y} \bigwedge_{i<m} \phi(\bar{t}_i(\bar{y}))$.

Let A be any model of T, and let $B = \langle a \rangle_A$, for some $a \in A$. $B \models T$ and so $B \models \exists\bar{x}\phi$. But every element of B is necessarily a term of a, so for some term $\bar{t}_0(a)$, $B \models \phi(\bar{t}_0(a))$. Suppose that for some choice of model, A_1, and element, a_1, $\langle a_1 \rangle_{A_1}$ does not have $t_0(a_1)$ satisfying ϕ. Then it must have some other term satisfying ϕ, $t_1(a_1)$. Repeat this until either every new choice leads to a repeated term, or we have exhausted all possible terms, and have a list $\{t_i(y) \mid i < |L|\}$ of terms which are possible satisfactions of ϕ. Then by compactness we can satisfy $T \cup \{\neg\phi(t_i(c)) \mid i < |L|\}$, which is impossible, since then the element named by c generates a substructure in which $\exists\phi$ is false.

8.1.1. Suppose λ is an infinite cardinal and A is a λ-saturated L-structure. Show that if E is an equivalence relation on n-tuples of elements of A which is first-order definable with parameters, then the number of equivalence classes of E is either finite or $\geq \lambda$.

Suppose there are exactly α equivalence classes of E. Let X be a set containing an element from each equivalence class, and consider the type $\{\neg xEy \mid y \in X\}$. If there are infinitely many equivalence classes of E, then by compactness this type can be completed. If $|X| < \lambda$, then by saturation this type must be realized, yielding an element not in the same equivalence class as any element of X, which is impossible by X's construction. Thus, $|X| \geq \lambda$, and there are therefore $\geq \lambda$ many equivalence classes.

8.1.2. Let A be an L-structure and λ a cardinal $> |A|$. Show that the following are equivalent. (a) A is λ-big. (b) A is λ-saturated. (c) A is finite.

(a) implies (b) by Theorem 8.1.2. To see that (b) implies (c), consider the set $\{x \neq a \mid a \in A\}$. If it is consistent, we can extend it to a type which is realized in A, since $|A| < \lambda$, but this is impossible. Thus, it is inconsistent. But if there are infinitely many elements in A, then any finite subset must be consistent, so there are only finitely many elements in A. For (c) implies (a), let $L^+ = L \cup \{R\}$, for R a new relation symbol, and let B be an L^+-structure with $(A, \bar{a}) \equiv (B, \bar{b})|L$, for $\bar{a}$ some sequence of length $< \lambda$. We must extend A to an L^+-structure. Since A is finite, B is finite, so there is a single sentence saying that the complete elementary diagrams of A and B are the same (with $\bar{a}$ mapping to $\bar{b}$). Thus, $(A, \bar{a}) = (B, \bar{b})|L$, and so we can just make $R^A = R^B$.

8.1.3. We define λ^- to be μ if λ is a successor cardinal μ^+, and λ otherwise. Show that if an L-structure A is not λ-saturated, then for all κ with $\max(|L|, \lambda^-) \leq \kappa < |A|$ there are elementary substructures of A of cardinality κ which are not λ-saturated.

If A is not λ-saturated, then there is some set X with $|X| < \lambda$ and $p \in S(X; A)$ such that p is not realized in A. $|X| \leq \lambda^-$. Now, for any such κ, just take an elementary substructure of A, B, containing X of size κ. Then B cannot realize p, since A does not.

8.1.4. (a) Show that if A is λ-big then A is strongly λ-homogeneous. (b) Show that if A is strongly λ-homogeneous, then A is λ-homogeneous. (c) Show that if A is $|A|$-homogeneous, then A is strongly $|A|$-homogeneous.

(a) Suppose we have $(A, \bar{a}) \equiv (A, \bar{b})$. By elementary amalgamation we can find an elementary extension of A, A_1 such that there is an elementary embedding, g_1, of A into A_1 such that $g_1(\bar{b}) = \bar{a}$. Now I claim $(A_1, A) \equiv (A_1, g_1(A))$. Given any sentence with parameters in A, $\varphi(\bar{c})$, $A_1 \models \varphi(\bar{c}) \Leftrightarrow A \models \varphi(\bar{c})$, since $A \preccurlyeq A_1$, and also $A_1 \models \varphi(g(\bar{c})) \Leftrightarrow g(A_1) \models \varphi(g_1(\bar{c}))$, since $g_1(A) \preccurlyeq A_1$. Finally, $A \models \varphi(\bar{c}) \Leftrightarrow g_1(A_1) \models \varphi(g_1(\bar{c}))$, since g_1 is an isomorphism of A to $g_1(A)$. Thus, $A_1 \models \varphi(\bar{c}) \Leftrightarrow A_1 \models \varphi(g_1(\bar{c}))$. Repeat this procedure with (A_1, A) and $(A_1, g_1(A))$ to get $A_2 \preccurlyeq A_3 \preccurlyeq \cdots$, and $g_2 \subseteq g_3 \subseteq \cdots$. Let $A' = \bigcup_{i<\omega} A_i$ and $g = \bigcup_{i<\omega} g_i$. Then g is an automorphism of A' taking $\bar{a}$ to $\bar{b}$, and $A \preccurlyeq A'$. Let $L^+ = L \cup \{R\}$, where R is a binary relation symbol, and let $R^{A'}(x, y) \Leftrightarrow g(x) = y$. $(A', \bar{a}, \bar{b})|L = (A, \bar{a}, \bar{b})$, so by λ-bigness of A, we can make $(A, \bar{a}, \bar{b})$ into a model of $\mathrm{Th}((A', \bar{a}, \bar{b}))$ with $(A', \bar{a}, \bar{b})$ considered as an L^+-structure. $(A', \bar{a}, \bar{b})$ as an L^+-structure says that R defines an automorphism taking $\bar{a}$ to $\bar{b}$, and so the expansion of A must have R defining an automorphism on A taking $\bar{a}$ to $\bar{b}$.

(b) Given $(A, \bar{a}) = (A, \bar{b})$, let g be an induced automorphism. For any c then, $(A, \bar{a}, c) \equiv (A, \bar{b}, g(c))$.

(c) If A is $|A|$-homogeneous, we can build an automorphism in stages, by induction: given $(A, \bar{a}) \equiv (A, \bar{b})$, list all the elements of A, $\langle a_i \mid i < |A| \rangle$, with $\bar{a}$ an initial segment of this list, and all the elements of A, $\langle b_i \mid i < |A| \rangle$, with $\bar{b}$ an initial segment. We define $\langle g_i \mid i < |A| \rangle$, a chain of functions, from the first sequence to the second, with g_i defined on all a_j, $j < i$, and $(A, \langle a_j \mid a_j \in \mathrm{dom}(g_i) \rangle) \equiv (A, \langle g(a_j) \mid a_j \in \mathrm{dom}(g_i) \rangle)$. g_0 maps a_i to b_i for $a_i \in \bar{a}$. At limit stages, take unions. At stage $i+1$, if we have not defined $g_{i+1}(a_i)$, we have $(A, \langle a_j \mid j < i \rangle) \equiv (A, \langle g_i(a_j) \mid j < i \rangle)$. By homogeneity, we can find b such that $(A, \langle a_j \mid j < i \rangle, a_i) \equiv (A, \langle g_i(a_j) \mid j < i \rangle, b)$. Let $g_{i+1}(a_i)$ be the b occurring first in the b_i's which has not been mapped to already. Now repeat this for b_i, finding the first a for which $g_i(a)$ is not defined, and mapping $g_{i+1}(a) = b_i$. Then $g = \bigcup_{i<|A|} g_i$ will be an automorphism.

8.1.5. Let λ be an infinite cardinal, A a λ-saturated structure and X a set of fewer than λ elements of A. Show (a) if an element a of A is not algebraic over X, then infinitely many elements of A realise $\mathrm{tp}_A(a/X)$, (b) if an element a of A is not definable over X, then at least two elements of A realize $\mathrm{tp}_A(a/X)$.

(a) Suppose there are exactly n realizations of $p(x) = \mathrm{tp}_A(a/X)$. Consider an $n+1$-type $q(x_0, \ldots, x_n)$ such that $q \supseteq p(x_0) \cup \ldots \cup p(x_n)$. If a is not algebraic, every finite subset of q is consistent, so q is realized by λ-saturation, but then there are $n+1$ realizations of p. Thus, a is not algebraic.

(b) Same procedure, with $n = 1$.

8.1.6. Show that if A is a λ-big L-structure and $\bar{a}$ is a sequence of fewer than λ elements of A, then $(A, \bar{a})$ is λ-big. Show that the same holds for λ-saturation, λ-homogeneity, and strong λ-homogeneity.

Suppose $(B, \bar{b})$ is such that $(B, \bar{b})|L \equiv ((A, \bar{a}), \bar{c})$, where $\bar{c}$ is a sequence of fewer than λ elements of A. Then $\bar{a}\bar{c}$ is a sequence of fewer than λ elements of A, so by λ-bigness for A, we can expand $(A, \bar{a}\bar{c})$ to a model of $\mathrm{Th}(B, \bar{b})$. This expansion expands $((A, \bar{a}), \bar{c})$ to a model of $\mathrm{Th}(B, \bar{b})$ as well. Thus $(A, \bar{a})$ is λ-big. Suppose X is a set with $|X| < \lambda$. Thus, if p is any complete type over X in $(A, \bar{a})$, then p can be written as a complete type over $X \cup \bar{a}$ in A, but $|X \cup \bar{a}| < \lambda$, so p has a realization in A which is a realization in $(A, \bar{a})$. Suppose $((A, \bar{a}), \bar{b}) \equiv ((A, \bar{a}), \bar{d})$ for some $\bar{b}$, $\bar{d}$ sequences of fewer than λ elements. Then $(A, \bar{a}\bar{b}) \equiv (A, \bar{a}\bar{d})$, so for any c we can find an e such that $(A, \bar{a}\bar{b}, c) \equiv (A, \bar{a}\bar{d}, e)$, but then $((A, \bar{a}), \bar{b}, c) \equiv ((A, \bar{a}), \bar{d}, e)$. Suppose $((A, \bar{a}), \bar{b}) \equiv ((A, \bar{a}), \bar{d})$, with the same properties. Then $(A, \bar{a}\bar{b}) \equiv (A, \bar{a}\bar{d})$, so by strong λ-homogeneity we can find an automorphism of A fixing $\bar{a}$ and sending $\bar{b}$ to $\bar{d}$, and so it is the required automorphism of $(A, \bar{a})$.

8.1.7*. Show that the following are equivalent, where λ, μ are any cardinals with $\min(\lambda, \omega) \leq \mu \leq \lambda^+$. (a) A is λ-saturated. (b) A is λ-homogeneous and μ-universal.

(a) implies (b) has been shown in the text. For (b) implies (a), it is clearly not true as stated, since then for any infinite model, λ-homogeneous would imply λ-saturation, as ω-universality is trivial for an infinite model. A counterexample to this is a language with κ many constants, and a structure A which contains just the interpretations of the constants. Then A is λ-homogeneous for every λ, but not λ-saturated for any λ. A is also κ-universal, since there are no models with cardinality $< \kappa$. Thus, κ-homogeneous and κ-universal need not imply κ-saturated (as claimed on page 214). Moreover, if $\kappa = \gamma^+$, then A is actually γ-homogeneous and γ^+-universal.

I prove instead that λ-homogeneous and λ^+-universal implies λ-saturated, for $|L| \leq \lambda$. Let A be λ-homogeneous and λ^+-universal, let X be any subset of A with $|X| < \lambda$, and let p be any complete type over X. We show that p is realized in A. p is realized in some elementary extension of A, B_1, and we can form an elementary substructure of this extension, containing X and p, B_2, which we can assume to have size λ. Then B_2 elementarily embeds into A, since A is λ^+-universal. Let g be the embedding. Since g is an isomorphism from B_2 to $g(B_2)$, and $g(B_2) \preccurlyeq A$, we have $(A, X) \equiv (B_1, X) \equiv (B_2, X) \equiv (g(B_2), g(X)) \equiv (A, g(X))$. But then let b be the realization of p in B_2. Then there is some a such that $(A, X, a) \equiv (A, g(X), g(b))$, by λ-homogeneity, but then a realizes p.

8.1.8. Show that if A and B are elementarily equivalent structures and A is ω-saturated, then B is ω-saturated if and only if it is back-and-forth equivalent to A.

Suppose B is ω-saturated. We show it is back-and-forth equivalent to A. We have $(A, \bar{a}) \equiv (B, \bar{b})$, and we wish to show that given $c \in A$, there is $d \in B$ such that $(A, \bar{a}, c) \equiv (B, \bar{b}, d)$. (This is enough by symmetry.) $\mathrm{tp}_A(\bar{a}, c/\emptyset)$ is realized in B, since B is ω-saturated, say by $\bar{b}', d'$. Then by ω-homogeneity (since B is ω-saturated), there is some d such that $(B, \bar{b}, d) \equiv (B, \bar{b}', d') \equiv (A, \bar{a}, c)$. For the converse, let $p(x)$ be any type over $\bar{b} = (b_0, \ldots, b_m)$ some tuple in B. Let $q(y_0, \ldots, y_m, x)$ be the type such that $q(b_0, \ldots, b_m, x) = p(x)$. Since A and B are back-and-forth equivalent, we can find $\bar{a}$ such that $(A, \bar{a}) \equiv (B, \bar{b})$. By ω-saturation of A, we can find $\bar{a}', c' \in A$ realizing q, and since $\mathrm{tp}_A(\bar{a}') = \mathrm{tp}_B(\bar{b}) = \mathrm{tp}_A(\bar{a})$, we can find $c \in A$ with $\bar{a}, c$ realizing q, so by back-and-forth, we can find d with $(A, \bar{a}, c) \equiv (B, \bar{b}, d)$, but then $q = \mathrm{tp}_A(\bar{a}, c) = \mathrm{tp}_B(\bar{b}, c)$, so $tp_B(c/\bar{b}) = p$.

8.1.9. Show that if $(A_i \mid i < \kappa)$ is an elementary chain of λ-saturated structures and $\mathrm{cf}(\kappa) \geq \lambda$ then $\bigcup_{i<\kappa} A_i$ is λ-saturated. Show the same for λ-homogeneity.

Let X be any set of size $< \lambda$ in $A = \bigcup_{i<\kappa} A_i$. Since $\mathrm{cf}(\kappa) \geq \lambda$, all of X is contained in some A_γ, $\gamma < \kappa$. Since A_γ is λ-saturated, every type over X is realized in A_γ, and hence in A.

Let $(A, \bar{a}) \equiv (A, \bar{b})$, where $\bar{a}$ and $\bar{b}$ are sequences of fewer than λ elements of A. By the same cofinality argument, we can find A_γ with both sequences contained in A_γ, and then use the λ-homogeneity of A_γ, along with the fact that $A_\gamma \preccurlyeq A$.

8.1.10. Show that the result of Exercise 9 fails for λ-bigness and strong λ-homogeneity.

Let A_0 be a model with 2 equivalence classes, E_1 and E_2, both with size ω. Given A_i, let A_{i+1} be an extension with an additional element in E_1. Let $A_\gamma = \bigcup_{i<\gamma} A_i$ for limits, and consider $A = \bigcup_{i<\omega_1} A_i$. Then each A_i is ω-big, since given an n-ary relation, R, and some theory T extending $\mathrm{Th}(A_i)$, we can find a countable model of T. This countable model must have two infinite equivalence classes, hence each has size ω, and so the natural isomorphism extends A_i to a model of T. However, A is not strongly 1-homogeneous, since, if $a \in E_1$ and $b \in E_2$, $(A, a) \equiv (A, b)$, but there is no automorphism taking a to b.

8.1.11. Show that if L and L^+ are languages with $L \subseteq L^+$ and A is a λ-big L^+-structure, then $A|L$ is a λ-big L-structure. Show the same for λ-saturation. On the other hand, show that if A is strongly λ-homogeneous, it need not follow that $A|L$ is λ-homogeneous.

For λ-big, apply Theorem 5.5.1. Let L' extend L by one new relation, and let B be an L'-structure, with $(B, \bar{b})|L \equiv (A, \bar{a})|L$, for some $\bar{a}$, $\bar{b}$ sequences of fewer than λ elements. We can assume $\bar{b} = \bar{a}$. Then Theorem 5.5.1 tells us that there is an $L' \cup L^+$-structure, D, which is an elementary extension of A, and into which B elementarily embeds, with a map g in which $g(\bar{a}) = \bar{a}$. Then by λ-bigness, we can make A into an $L^+ \cup L'$-structure, A', such that $\mathrm{Th}(A') = \mathrm{Th}(D) \supseteq \mathrm{Th}(g(B)) = \mathrm{Th}(B)$. Then $A'|L'$ gives an expansion of $A|L$, showing it is λ-big.

Suppose p is some type in L over X a set of $< \lambda$ elements of A. We can extend p to a complete type over L^+, and so the extension is realized in A, and thus p is too.

Let A be a model with 2 equivalence classes, and a relation, R, which picks out one of the classes.

Let L be the language with just the equivalence relation. Let one class have ω elements, and one have ω_1. Then A is strongly $|A|$-homogeneous, but $A|L$ is not $|A|$-homogeneous.

8.1.12. Show that if L, L^+ are first-order languages with $L \subseteq L^+$, A is an L^+-structure, $A|L$ is λ-big, and A is a definitional expansion of $A|L$, then A is λ-big. Show that the same holds for λ-saturation, λ-homogeneity and strong λ-homogeneity.

Let L' be L^+ together with a new relation, R, and let B be an L'-structure such that $(B, \bar{b})|L^+ \equiv (A, \bar{a})$ for some sequences $\bar{b}$ and $\bar{a}$ of fewer than λ elements. Since A is a definitional expansion of $A|L$, and the property of being a definitional expansion is first-order, $(B, \bar{b})|L^+$ is a definitional expansion of $(B, \bar{b})|L$ with precisely the same definitions. Let $L'' = L \cup \{R\}$. Then we can expand $A|L$ to be an L''-structure such that $(A, \bar{a}) \equiv (B, \bar{b})|L''$, since $A|L$ is λ-big. Then, using the same definitions, $A|L''$ can be expanded to be an L'-structure. Any sentence in $\mathrm{Th}(A|L')$ is equivalent (in $A|L'$) to one in L'', and hence the corresponding sentence is true in $B|L''$, and therefore is in $\mathrm{Th}(B)$, and vice versa.

If p is a type over a set X with $|X| < \lambda$ in A, then p can be rewritten as a type in $A|L$, and so it is realized.

If $(A, \bar{a}) \equiv (A, \bar{b})$, with $\bar{a}$ and $\bar{b}$ sequences of fewer than λ elements in A, given c, we can find $d \in A$ such that $\mathrm{tp}_{A|L}(\bar{a}c) = \mathrm{tp}_{A|L}(\bar{b}, d)$, but since A is a definitional expansion, the type in $A|L$ uniquely determines the type in A, so d works.

With the same starting conditions, we can find an automorphism, g, of $A|L$ such that $g(\bar{a}) = \bar{b}$. Since every formula of L^+ is equivalent to one in L, g must be an automorphism of A.

8.1.13*. Let L be a first-order language. Show that if A is a λ-big L-structure and $\phi(x)$ is a formula of L such that $\phi(A)$ is the domain of a substructure B of A, then B is λ-big. Show that the same holds for λ-saturation, λ-homogeneity and strong λ-homogeneity.

Suppose L^+ is a language extending L, and $\mathfrak{D}$ is an L^+-structure, with $(\mathfrak{D}, \bar{d})|L \equiv (B, \bar{b})$, for some sequences of fewer than λ elements. Let $\mathrm{Th}_\phi(\mathfrak{D})$ denote the theory of $\mathfrak{D}$ relativized to ϕ (so every quantifier is of the form $\forall x \in \phi$ or $\exists x \in \phi$). Consider $T = \mathrm{Th}(A) \cup \mathrm{Th}_\phi(\mathfrak{D})$. By Theorem 5.5.1 (the extension of Craig's Interpolation Theorem), if T is not consistent, there is some $\psi \in L$ such that $\mathrm{Th}(A) \vdash \psi$ and $\mathrm{Th}_\phi(\mathfrak{D}) \vdash \neg\psi$. But $\mathrm{Th}_\phi(\mathfrak{D}|L) = \mathrm{Th}_\phi(B) \subseteq \mathrm{Th}(A)$, so this is impossible. Thus, T is consistent, and so we can find some $\mathfrak{E}$ realizing it. Then A can be extended to a model of $\mathrm{Th}(\mathfrak{E})$, which extends B to a model of $\mathrm{Th}(\phi(E)) = \mathrm{Th}(\mathfrak{D})$.

Suppose p is a type in B over X, a set of fewer than λ elements in B. Relativizing p, we can extend it to a type in A, q. q is realized in A, and q has the formula $\phi(x)$, so the realization is in B.

The final two claims are false. Let A be a model with 2 equivalence classes, E_1 and E_2, each with ω_1 elements, a unary relation, R, which is true on all of E_1 and ω elements of E_2, and a binary relation S with $S(a, b)$ true for a unique $a \in E_2$, with $\neg R(a)$, and every $b \in E_2$. Then A is strongly ω_1-homogeneous, since the full type of an element x is given by the truth of the formulas $R(x)$, $\exists y(S(y, x))$,

$\exists y(S(x,y))$. Let B be the substructure $R(A)$. Then S is always false on B, and B is just the model with two equivalence classes, one of size ω and one of size ω_1, and so is not ω_1-homogeneous.

8.1.14*. Show that if A and B are λ-big structures, then the disjoint sum of A and B is λ-big. Show the same for λ-saturation, λ-homogeneity and strong λ-homogeneity.

The statement is false for λ-big. Let $A = \omega$, with no structure, let $B = \omega_1$, with no structure. A and B are clearly ω-big. Let $\mathfrak{D}$ have universe $\omega_1 + \omega_1$, and a relation R which puts the ω_1's in bijection with each other. Then $\mathfrak{D}$ is a counterexample to $A + B$ being ω-big.

Given a type $p(x)$ over $X \subset A + B$, $|X| < \lambda$, p must determine if x lies in A or B. Then the type is equivalent to the restriction of p to A (or B). But since A (B)has a realization of this restriction, p is realized in $A + B$. $A + B$ is certainly λ-homogeneous if A and B are, since given a sequence, the type of a new element is completely determined by its type over the elements in the universe it came from, and so we can find an element from that universe of the proper type by λ-homogeneity. Strong λ-homogeneity follows similarly, noting that automorphisms of A and B extend to automorphisms of $A + B$.

8.1.15*. Let L be a first-order language and A a saturated L-structure of cardinality $> |L|$. Show that A is $|A|$-big.

Let $\lambda = |A|$. Note that we need only show that A is splendid, since $(A, \bar{a})$ is λ-saturated if $\bar{a}$ has fewer than λ elements, and so the conclusion will hold for it as well. Let $L^+ = L \cup \{R\}$, for some new relation R, and let B be an L^+-structure with $B|L \equiv A$. We can assume $|B| = \lambda$. Let $\langle b_i \mid i < \lambda \rangle$ enumerate B. Build a chain of structures, $\langle B_i \mid i < \lambda \rangle$. First, let B_0 be an elementary substructure of B with $|B_0| = |L|$. For each i, let B_i be an elementary substructure of B extending every B_j, $j < i$, and containing the first b_k not in $\bigcup_{j<i} B_j$. Then $|B_i| = |L| + |i|$. As well, $B = \bigcup_{i<\lambda} B_i$. By $|A|^+$-universality, $B_0|L$ elementarily embeds into A, and hence maps onto an elementary substructure of A of cardinality $|B_0|$, C_0. Let f_0 be the embedding. Then f_0 induces a structure on C_0, making it into an L^+-structure. We construct a chain $\langle C_i \mid i < \lambda \rangle$ of L^+-structures, $|C_i| = |B_i|$, with a chain of elementary embeddings $f_i : B_i \to C_i$, and elementary substructures of A, $A_i = C_i|L$, such that $A = \bigcup_{i<\lambda} A_i$. C_0 has been defined. Fix a well-ordering of A, like that of B above. We define C_{i+1}. Let C' be an amalgam of C_i and B_{i+1} over B_i, in other words, $C_i \preccurlyeq C'$, and there is an elementary embedding $g : B_{i+1} \to C'$ with $g(B_i) = f_i(B_i) = C_i$, pointwise. We can do this because the elementary diagrams of B_{i+1} and C_i are compatible. Now find an amalgam, C^*, of C' and A over C_i such that C^* is an L^+-structure, with $A \preccurlyeq C^*$ (as L-structures), and an elementary embedding $h : C' \to C^*$ with $h(C_i) = C_i$, pointwise. Let a be the first element in the well-ordering of A fixed above which is not in C_i. We can take an elementary substructure of C^* of size $|B_{i+1}|$ which contains $h(g(B_{i+1}))$ and a, C'_{i+1}. Let $D = C'_{i+1} \cap A$. We know $(C'_{i+1}, D)|L \equiv (A, D)$. Through an easy saturation argument, we can elementarily embed C'_{i+1} in A as an L-structure, fixing D, say by a map e. e induces an L^+-structure on $C_{i+1} = e(C'_{i+1})$ as before. Then ehg elementarily embeds B_{i+1} in C_{i+1} and all required properties are satisfied. At limits there is nothing to do, and the final induced map gives a surjective elementary embedding of B to A, which is therefore an isomorphism, and thus induces an L^+-structure on A.

8.1.16. (a) Show that if L is a first-order language and $\lambda > |L|$, then an L-structure A is λ-saturated if and only if it is λ-compact. (b) Show that every structure is ω-compact. (c) Show that for every infinite cardinal λ, a λ-saturated structure is λ-compact. (d) Give an example of an $|L|$-compact L-structure which is not L-saturated.

(c) and forward direction of (a): Let A be λ-saturated. Let p be a partial type over X, $|p| < \lambda$. p mentions fewer than λ many elements of X, so we can consider p as a partial type over X', with $|X'| < \lambda$. Then any completion of p over X' has a realization in A, since A is λ-saturated.

(a) backwards: Let A be λ-compact, and let p be a complete type over X, with $|X| < \lambda$. Then $|p| < |X| + |L| < \lambda$, so p has a realization.

(b): A type of size $< \omega$ is a finite set of formulas consistent with $\mathrm{Th}(A)$. There must be a witness to their consistency in A, since $\mathrm{Th}(A)$ must say $\exists x(\ldots)$.

(d): If $|L| = \omega$, then every L-structure is $|L|$-compact, so just choose a non-saturated one. Let L consist of countably many distinct constants, and let A be the model consisting of just those constants. Then A is not $|L|$-saturated.

8.1.17. Suppose the L-structure A is λ-compact, $\bar{a}$ is a tuple from A, $\psi(\bar{x}, \bar{y})$ is a formula of L and $\Phi(\bar{x}, \bar{y})$ is a set of fewer than λ formulas of L. Show that if $A \models \forall \bar{x}(\bigwedge \Phi(\bar{x}, \bar{a}) \to \psi(\bar{x}, \bar{a}))$, then there is a finite subset Φ_0 of Φ such that $A \models \forall \bar{x}(\bigwedge \Phi_0(\bar{x}, \bar{a}) \to \psi(\bar{x}, \bar{a}))$.

Suppose the then-clause is false. Then for every finite subset, there is a counterexample. Consider the set of formulas $\{\Phi_0(\bar{x}, \bar{a}) \wedge \neg\psi(\bar{x}, \bar{a}) \mid \Phi_0 \subseteq \Phi \wedge |\Phi_0| < \omega\}$. By λ-compactness, there is a witness to this in A, which then contradicts the if-clause.

8.1.18. Let L be a countable first-order language and A a countable atomic L-structure. Show that A is homogeneous.

A realizes only principal types. Thus, if $(A, \bar{a}) \equiv (A, \bar{b})$, and some c is given, c satisfies some $\varphi(x, \bar{a})$ which generates the type of c over $\bar{a}$. But $\exists x \varphi(x, \bar{b})$, since $(A, \bar{a}) \equiv (A, \bar{b})$, so we can find d such that $(A, \bar{a}, c) \equiv (A, \bar{b}, d)$.

8.1.19. Let L and L^+ be first-order languages with $L \subseteq L^+$. Let λ be an infinite cardinal such that the number of symbols in the signature of L^+ but not in L is less than λ. Show that if A is an L^+-structure and B is a λ-big L-structure such that $A|L \equiv B$, then B can be expanded to a structure $B' \equiv A$.

We may certainly assume that all new symbols are relations, say $\langle R_i \mid i < \gamma < \lambda \rangle$. Add a binary function to L^+, f, which maps $A \times A$ injectively to A. Now, for any n-ary relation $R_i \in L^+$, we can replace it by a unary relation R'_i by defining $R'_i(x) \Leftrightarrow x = f(y_0, f(y_1, f(\ldots, f(y_{n-2}, y_{n-1})))\ldots) \wedge$

$R_i(y_0, \ldots, y_{n-1})$. Thus, we can assume that $L^+ \setminus L$ has only unary relations and one binary function. Now replace these by one quaternary binary relation, S, and γ many constant symbols $\langle c_i \mid i < \gamma \rangle$, with the following properties: $S(c_0, x_0, x_1, x_2) \Leftrightarrow f(x_0, x_1) = x_2$, and $S(c_i, x_0, x_1, x_2) \Leftrightarrow R'_i(x_0)$. Since there are fewer than λ many constant symbols, and B is λ-big, we can enumerate the interpretations of the constant symbols in A, and then find an expansion of B elementarily equivalent to A.

8.2.1. (a) Let T be a countable first-order theory with infinite models. Show that T has a countable strongly ω-homogeneous model. (b) Show that if the continuum hypothesis fails, then there is a countable first-order theory with infinite models but with no strongly ω_1-homogeneous model of cardinality ω_1.

(a) By Corollary 8.2.6, if A is a countable model of T, then there is an elementary extension of A which is countable and strongly ω-homogeneous.

(b) Let T be the theory of a perfect tree in ω. Then any model of T with cardinality ω_1, A, must have at least one "branch," but not every chain of nodes can define a branch, since there are 2^ω of them. Consider a countable chain of nodes with no branch at the end, H, and a countable chain with a branch at the end, K. $(A, K) \equiv (A, H)$ by a back-and-forth argument, but there is no automorphism extending this equivalence, because the branch at the end of K cannot be mapped to anything.

8.2.2*. Let T be a complete theory in a countable first-order language. Suppose T has infinite models, and there is a finite set of types of T, such that all countable models of T realizing these types are isomorphic. Show that T is ω-categorical.

Let $p_1(\bar{x}_1), \ldots, p_n(\bar{x}_n)$ be the specified types. Adjoin new constants, $\bar{a}_1, \ldots, \bar{a}_n$ to the language and let $T' = T \cup \{p_i(\bar{a}_i) \mid i \leq n\}$. By compactness, T' is consistent. Moreover, it is ω-categorical. But then for every n, $S_n(T')$ is finite. Since each type in $S_n(T)$ extends to at least one type in $S_n(T')$, this implies that $S_n(T)$ is finite, and hence that T is ω-categorical.

8.2.3. Show that if T is a countable complete first-order theory, then the number of countable models of T, counted up to isomorphism, is not 2.

Suppose T is not ω-categorical. Then it has a non-principal type $p(\bar{x})$. Suppose T had exactly 2 countable models. Then, since we have at least 2 from realizing $p(\bar{x})$ and omitting $p(\bar{x})$, those are all of them. Thus, all countable models of T realizing $p(\bar{x})$ are isomorphic, but then T is ω-categorical, so actually T does not have 2 countable models.

8.2.4. Show that if $2 < n < \omega$ then there is a countable complete first-order theory T such that up to isomorphism, T has exactly n countable models.

Start with the theory of a dense linear ordering without endpoints. Add countably many constants, $\langle c_i \mid i < \omega \rangle$, with $c_i < c_j$ $(i < j)$. There are then three non-isomorphic models, all with universe $\mathbb{Q}$, determined by whether the c_i's are unbounded, have no sup in $\mathbb{Q}$, or have a sup in $\mathbb{Q}$. To get $n > 3$, add a "coloring" of $\mathbb{Q}$ with $n-2$ colors, such that each color is dense in $\mathbb{Q}$. Then the non-isomorphic models are the first two, along with one for a sup of each possible color. Back-and-forth arguments show in all these cases that these are the only non-isomorphic models, and also that these models are non-isomorphic.

8.2.5. Show that Theorem 8.2.5 holds with 'strongly λ-homogeneous' in place of 'λ-homogeneous.'

We reproduce the proof. By Theorem 8.2.1 we have a λ-big elementary extension B of A. Write $\nu = (|A| + |L|)^{<\lambda}$, noting that $\nu \geq \lambda$. For every $\bar{a}, \bar{b}$ sequences of length $< \lambda$ such that $(B, \bar{a}) \equiv (B, \bar{b})$, fix an automorphism g taking $\bar{a}$ to $\bar{b}$. For D any elementary substructure of B of cardinality at most ν, we can find a structure D^* with $D \preccurlyeq D^* \preccurlyeq B$ so that if $\bar{a}$ and $\bar{b}$ are two sequences of elements of D, both of length $< \lambda$, and $(D, \bar{a}) \equiv (D, \bar{b})$, then there is an elementary embedding of D in D^* such that $\bar{a}$ goes to $\bar{b}$. We can find D^* as the union of a chain of elementary substructures of B, $\langle D_i \mid i < \nu \rangle$. Consider $(\bar{a}, \bar{b})$ such that $(D, \bar{a}) \equiv (D, \bar{b})$. The number of pairs is at most $\nu^{<\lambda} = \nu$. For each such pair, we have a specific automorphism of B chosen above, g, taking $\bar{a}$ to $\bar{b}$. List the $\leq \nu$ automorphisms as $\langle g_i \mid i < \nu \rangle$ such that every automorphism appears cofinally often in the sequence. Define $D_0 = D$, and limits are unions. With D_i already defined, set D_{i+1} to be an elementary extension of D_i inside B containing $g_i(D_i)$ of size ν. Let $D^* = \bigcup_{i<\nu} D_i$. D^* has size ν, and for any g_i, $g_i(D^*) = D^*$, since for any element $a \in D^*$, $a \in D_j$ for some $j < \nu$, and then a copy of g_i appears above j, so $g_i(a) \in D^*$.

Now we build a chain $\langle A_i \mid i < \lambda \rangle$ of elementary substructures of B, so that for each $i < \lambda$, $A_{i+1} = A_i^*$. $A_0 = A$, and at limits, take unions. Let $C = \bigcup_{i<\lambda} A_i$. Then C has cardinality at most $\lambda \cdot \nu = \nu$. As well, it is strongly λ-homogeneous, since given any $\bar{a}, \bar{b}$ with length $< \lambda$, since λ is cofinal we can find A_i containing both. Let g be the automorphism selected at the beginning for this $\bar{a}$ and $\bar{b}$. Let c be any element in C. Then $c \in A_j$ for some $j \geq i$. Then $g(c) \in A_{j+1}$. Thus g maps C automorphically to itself.

8.2.6*. Let L be a first-order language, T an $\forall_2$ theory in L, A an L-structure which is a model of T and λ a regular cardinal $> |L|$. Show that there is an e.c. model B of T such that $A \subseteq B$, $|B| \leq |A|^{<\lambda}$ and for every sequence $\bar{b}$ of $< \lambda$ elements of B, every e.c. model C of T extending B and every element c of C, there is an element d of B such that $(B, \bar{b}, d) \equiv_1 (C\bar{b}, c)$.

We construct B in stages. Let $\kappa = |A|^{<\lambda}$. List as $(p_i(x, \bar{y}), \bar{a}_i)_{i<\kappa}$ all pairs $(p, \bar{a})$ where p is a partial type containing only $\exists_1 \cup \forall_1$ formulas of L, and $\bar{a}$ is a sequence in A of length $< \lambda$. By induction on i, define a chain of structures $(A_i \mid i < \kappa)$ by: $A_0 = A$. For limit ordinals, take unions. At successors, if $p_i(x, \bar{a})$ is realized for some $C \models T$, with $C \supseteq A_i$, choose C minimal with this property and set $A_{i+1} = C$, and $A_{i+1} = A_i$ otherwise. Note that we can ensure that $|C| = |A_i|$, since if C is too large, take an elementary substructure realizing p_i and containing A_i of size $|A_i|$. Define $A^* = \bigcup_{i<\kappa} A_i$. Then $|A^*| \leq |A| \cdot \kappa = \kappa$. Now set $A^{(0)} = A$, $A^{(i+1)} = {A^{(i)}}^*$, and take limits at unions. Put $B = \bigcup_{i<\lambda} A^{(i)}$. $B \models T$, since T is $\forall_2$, and B is the union of several chains. B is e.c., since this construction includes the construction of Theorem 7.2.1. $|B| \leq \lambda \cdot \kappa = \kappa$. Finally, let C be any model containing B. Then, since B is e.c., for any $\bar{b}$ a sequence of fewer than λ elements in B, $(B, \bar{b}) \equiv_1 (C, \bar{b})$. Let c be any element of C. Let p be the $\exists_1 \cup \forall_1$ type of c over $\bar{b}$. Since $\bar{b}$ has fewer than λ elements and λ is regular, $\bar{b}$ is in $A^{(i)}$ for some i. Then $p(x, \bar{b})$ is realized in $A^{(i+1)}$, by some d, so then $(B, \bar{b}, d) \equiv_1 (C, \bar{b}, c)$.

8.2.7**. Let T be an $\forall_2$ theory in a first-order language L. (a) Show that every model of T can be embedded in some infinite-generic model of T. (b) Show that if A and B are infinite-generic models of T with $A \subseteq B$ then $A \preccurlyeq B$ (c) Show that if T is companionable, then the infinite-generic models of T are exactly the e.c. models.

(a) Every model of T can be embedded in a model like B above, for some $\lambda \geq \omega$. B is existentially universal, and thus infinite-generic.

(b) The key point will be that if M is an existentially universal (e.u.) model, and $(M, \bar{a}, \bar{b}) \equiv_1 (M, \bar{a}, \bar{c})$, then for any $d \in M$, there is an e such that $(M, \bar{a}, \bar{b}, d) \equiv_1 (M, \bar{a}, \bar{c}, e)$. To show this, suppose we have elements as described, and let $p(x, \bar{a}, \bar{b})$ be the $\exists_1 \cup \forall_1$ type of d over $\bar{a}\bar{b}$. I claim that $p(x, \bar{a}, \bar{c})$ is consistent with $T \cup \text{diag}(M)$. Suppose not. By compactness, we can find $\theta(x, \bar{a}, \bar{c}) \in p$ and quantifier-free $\psi(\bar{a}, \bar{c}, \bar{e}) \in \text{diag}(M)$, with $T \vdash \psi(\bar{a}, \bar{c}, \bar{e}) \to \forall x \neg\theta(x, \bar{a}, \bar{c})$. Then $T \vdash \forall \bar{y}\bar{z}\bar{w}(\psi(\bar{y}, \bar{z}, \bar{w}) \to \forall x \neg\theta(x, \bar{y}, \bar{z}))$. But since $(M, \bar{a}, \bar{b}) \equiv_1 (M, \bar{a}, \bar{c})$, we can find $\bar{e}'$ such that $\psi(\bar{a}, \bar{b}, \bar{e}')$, so $M \models \forall x \neg\theta(x, \bar{a}, \bar{b})$, which is impossible. Thus, $p(x, \bar{a}, \bar{c})$ is consistent with $T \cup \text{diag}(M)$. Hence, we can find a model of T, C, containing M and realizing p, say by f'. Since M is e.u., we can find $f \in M$ such that $(M, \bar{a}, \bar{c}, f) \equiv_1 (C, \bar{a}, \bar{c}, f')$. Thus $p(f, \bar{a}, \bar{c})$, and so $(M, \bar{a}, \bar{b}, d) \equiv_1 (M, \bar{a}, \bar{c}, f)$.

Now, let $A \subseteq B$ be infinite-generic models. Note that we can assume that B is e.u., since there is some N e.u. with $B \preccurlyeq N$, and so if $A \preccurlyeq N$, then $A \preccurlyeq B$. We show that if we have $(A, \bar{a}) \equiv_1 (B, \bar{b})$, ($\bar{a}$ a tuple in A, $\bar{b}$ a tuple in B), then for any c in B (A), we can find d in A (B) such that $\equiv_1$ still holds. $(A, \bar{a}) \equiv_1 (B, \bar{a})$ through a compactness argument, since $A \preccurlyeq M$, with M e.u. Thus $(B, \bar{b}) \equiv_1 (B, \bar{a})$. Thus, given $c \in B$, we can find $c' \in B$ such that $(B, \bar{b}, c) \equiv_1 (B, \bar{a}, c')$. If A is e.u. as well, then we can find d such that $(A, \bar{a}, d) \equiv_1 (B, \bar{a}, c')$, and thus $(A, \bar{a}, d) \equiv_1 (B, \bar{b}, c)$. If c is an element of A, then $(A, \bar{a}, c) \equiv_1 (B, \bar{a}, c)$, and so we can find d in B with $(B, \bar{b}, d) \equiv_1 (B, \bar{a}, c) \equiv_1 (A, \bar{a}, c)$. If A is not e.u., I don't know.

(c) Let A be infinite-generic, and let $B \supset A$ be any model of T containing A. B can be embedded in an infinite-generic model of T, C, and $A \preccurlyeq_1 C$, so any existential formula with parameters in A and a witness in B (and thus a witness in C), has a witness in A. Thus, A is existentially closed. If A is existentially closed, we can embed A into an infinite-generic, B. But then, since the infinite-generic is existentially closed, and the theory of e.c. structures of T is model complete, $A \preccurlyeq B$, and so if M is the existentially universal model of which B is an elementary substructure, $A \preccurlyeq M$, and thus A is infinite-generic.

8.3.1. Let T be a theory in a first-order language L and $\Phi(\bar{x})$ a set of formulas of L such that if $f : A \to B$ is any surjective homomorphism between models A, B of T and $A \models \bigwedge \Phi(\bar{a})$ then $B \models \bigwedge \Phi(\bar{b})$. Show that Φ is equivalent modulo T to a set $\Psi(\bar{x})$ of positive formulas of L.

We prove Theorem 8.3.3(b) with ϕ replaced by Φ. Let Ψ be the set of all consequences of Φ in which every relation symbol in Σ is positive. Let the number of free variables in Φ be $< \lambda$, with $\lambda \geq |L|$. We show that every model of $\bigwedge \Psi(\bar{a})$ is a model of $\bigwedge \Phi(\bar{a})$. Let B be any model of T with $B \models \bigwedge \Psi(\bar{b})$. We can assume B is λ-saturated by taking an elementary extension, (which preserves whether or not $\bigwedge \Phi(\bar{b})$). Now, if Θ is the set of all formulas in L in which every relation symbol in Σ is positive, let $\Theta_B = \{\theta \mid \theta \in \Theta \wedge \theta(\bar{b}) \in \mathrm{Th}(B)\}$ and let $\bar{\Theta}(\bar{b}) = \{\neg\theta(\bar{b}) \mid \theta \notin \Theta_B\}$. I claim that $T \cup \Phi(\bar{a}) \cup \bar{\Theta}(\bar{a})$ has a model, for $\bar{a}$ a sequence of new constants. If not, then $T \vdash \phi(\bar{a}) \to (\theta_1(\bar{a}) \vee \ldots \vee \theta_n(\bar{a}))$, for some $\phi \in \Phi$, $\theta_i \in \bar{\Theta}$, but the disjunction of positive formulas is positive, so it lies in Ψ, as a consequence of Φ, and thus is true in B, so the θ_i's cannot all be in $\bar{\Theta}$. Thus, the theory is consistent, and we can find some λ-saturated model, A. $(A, \bar{a}) \Rrightarrow_\Theta (B, \bar{b})$, and both are still λ-saturated, since there are fewer than λ elements in the sequences, so by the lemma, there is a surjective homomorphism from some elementary substructure of $(A, \bar{a})$ to some elementary substructure of $(B, \bar{b})$. In the first elementary substructure, $\bigwedge \Phi(\bar{a})$, and so in the second, $\bigwedge \Phi(\bar{b})$, and thus in B.

8.3.2. Let L be a first-order language and Θ a set of formulas of L. If A, B are L-structures, we write $(A, \bar{a}) \Rrightarrow_\Theta (B, \bar{b})$ to mean that for every formula θ in Θ, if $A \models \theta(\bar{a})$ then $B \models \theta(\bar{b})$. Show (a) if Θ is closed under conjunction, disjunction, and both existential and universal quantification, and A and B are λ-saturated L-structures with $\lambda \geq |L|$ such that $A \Rrightarrow_\Theta B$, then there are sequences $\bar{a}$, $\bar{b}$ in A, B respectively, both of length λ, such that $\bar{a}$, $\bar{b}$ list the domains of elementary substructures A', B' of A, B respectively, and $(A, \bar{a}) \Rrightarrow_\Theta (B, \bar{b})$. (b) If Θ is closed under conjunction and existential quantification, and A and B are L-structures such that $A \Rrightarrow_\Theta B$ and B is $|A|$-saturated, then there are sequences $\bar{a}$, $\bar{b}$ in A, B respectively such that $\bar{a}$ lists the domain of A and $(A, \bar{a}) \Rrightarrow_\Theta (B, \bar{b})$.

(a) We build up $\bar{a}$ and $\bar{b}$ as in Lemma 8.3.4, such that for each $i \leq \lambda$, $(A, \bar{a}|i) \Rrightarrow_\Theta (B, \bar{b}|i)$, and $\bar{a}$ and $\bar{b}$ are domains of elementary substructures of A and B, respectively. Use an inductive game between players $\exists$ and $\forall$. First we show that $\exists$ can always preserve the first condition. At the beginning, $A \Rrightarrow_\Theta B$. At limits, there is nothing to do. So all we need to consider is the successor step, $i = j + 1$. We have $\bar{a}$ and $\bar{b}$, both of length j, with $(A, \bar{a}) \Rrightarrow_\Theta (B, \bar{b})$. We wish to extend them. Let $\forall$ choose an element in A, a_j. Let $\Phi(\bar{x}, y)$ be the set of all formulas $\phi(\bar{x}, y)$ in Θ such that $A \models \phi(\bar{a}, a_j)$. Thus, for any finite subset of Φ, $\phi_0, \ldots, \phi_{n-1}$, we have $A \models \exists y \bigwedge_{i<n} \phi(\bar{a}, y)$. Since Φ is closed under conjunctions and existential quantification, this formula is in Φ, and so, by inductive assumption, $B \models \exists y \bigwedge_{i<n} \phi(\bar{b}, y)$ for every finite subset of Φ. But then, since B is λ-saturated, we can find b_j realizing this type, and this extends $\bar{b}$ appropriately.

If $\forall$ chooses an element in B, b_j, then let $\Phi'(\bar{x}, y)$ be the set of all formulas $\neg\phi(\bar{x}, y)$, with $\phi(\bar{x}, y) \in \Theta$, such that $B \models \neg\phi(\bar{b}, b_j)$. Form finite conjunctions as before. The negation of each conjunction is a disjunction of universal quantifications of formulas of Θ, hence the negations are false in A, and so the conjunctions are true. Thus we can realize the type in A, by a_j.

Now, in this process let $\forall$ make choices as follows: for every formula $\phi(\bar{a}|j, y)$ such that $A \models \exists y \phi(\bar{a}|j, y)$, at some point later on choose a witness, and likewise for B. Then by the Tarski-Vaught criterion, $\bar{a}$ and $\bar{b}$ will be elementary substructures of A and B, so we are done.

(b) Modify the above process so that $\forall$ chooses only elements of A, to exhaustion. Note that all we needed were conjunctions and existential quantification for that part of the proof.

8.3.3. Let L be a first-order language containing a 1-ary relation symbol P. Let A and B be elementarily equivalent L-structures of cardinality λ. Suppose that B is saturated, but make the following weaker assumption on A: if X is any set of fewer than λ elements of A and $\Phi(x)$ is a type over X with respect to A, which contains the formula $P(x)$, then Φ is realized in A. Show that there is an elementary embedding $f : A \to B$ which is a bijection from P^A to P^B.

We use the traditional construction by games. List the elements of A as $\langle a_i \mid i < \lambda \rangle$, and of $P(B)$ as $\langle b_i \mid i < \lambda \rangle$ (if $P(B)$ is finite, this is trivial, and otherwise, it has size λ by saturation). Define a chain of maps, $\langle f_i \mid i < \lambda \rangle$. f_0 can be the empty map. At stage $i+1$, let a be the first element not yet mapped (not in $\operatorname{dom}(f_i)$). $\operatorname{tp}_A(a/\operatorname{dom}(f_i))$ translates to a type in B over $f_i(\operatorname{dom}(f_i))$. Then this type is realized by some b by λ-saturation. Let $f_{i+1}(a) = b$. Also, let b' be the first element of $P(B)$ not mapped to yet. Perform the same procedure backwards to get a' in A. Define $f_{i+1}(a') = b'$. At limits, take unions. Then $\bigcup_{i<\lambda} f_i$ embeds A into B and maps $P(A)$ bijectively onto $P(B)$.

8.3.4. Let L be a first-order language, let A and B be $|L|$-saturated L-structures, and suppose that every sentence of $\operatorname{Th}(A)$ which is either positive or $\forall_1$ is in $\operatorname{Th}(B)$. Show that there are elementary substructures A', B' of A, B respectively, a surjective homomorphism $f : A' \to B'$ and an embedding $e : B' \to A$.

By Exercise 2, we can find A', B' such that there is a surjective homomorphism mapping A' to B'. B' can be embedded in A because $B' \Rrightarrow_1 A$, by Theorem 8.3.1.

8.3.5. Let L be a first-order language and T a theory in L. Show that the following are equivalent, for every sentence ϕ of L. (a) For every model A of T and endomorphism $e : A \to A$, if ϕ is true in A then ϕ is true in the image of A. (b) ϕ is equivalent modulo T to a positive boolean combination of positive sentences of L and $\forall_1$ sentences of L.

For (a) implies (b), assume (b) is false. Then let B be a model with every positive and $\forall_1$ consequence of T and ϕ true, but not ϕ. By the usual arguments, we can find a model, A, in which ϕ is true, but no extra positive or $\forall_1$ consequences are. Then the endomorphism defined by composing the two maps above maps A to a model in which ϕ is false, since the second map embeds an elementary substructure of B. Thus, (a) is false.

For (b) implies (a), let $\psi_1, \ldots, \psi_n$ be positive/$\forall_1$ sentences which can be combined to yield ϕ. Let e be any endomorphism of A. e is surjective on its image, and so e preserves all of the positive sentences among the ψ's. As well, since $e(A) \subseteq A$, $\forall_1$ sentences are preserved. Thus, ϕ is true in $e(A)$ iff ϕ is true in A.

8.3.6*. Let L be a first-order language whose symbols include a 2-ary relation symbol $<$, and let T be a theory in L and $\phi(\bar{x})$ a formula of L. In the terminology of Exercise 2.4.5, show that the following are equivalent. (a) If A and B are models of T and A is an end-extension of B, then for every tuple $\bar{b}$ of elements of B, $B \models \phi(\bar{b})$ implies $A \models \phi(\bar{a})$. (b) ϕ is equivalent modulo T to a Σ_1^0 formula $\psi(\bar{x})$ of L.

(a) implies (b): We reverse the notation, so B is an extension of A. If B is an end-extension of A, then for every a in A, if $A \models c < a$, then c is in A. Note that Σ_1^0 formulas are closed under existential quantification, conjunctions and disjunctions. By adjoining constants, we can assume that ϕ is a sentence. Let Ψ be all of the $\exists_1$ consequences of ϕ, and let B_0 be a model of T with $B \models \bigwedge \Psi$. Let $\lambda > |L|$ be a regular cardinal. Let B be a λ-saturated elementary extension of B_0. By the usual compactness argument, we can find A_0, a model of T and ψ, such that no $\exists_1$ sentence not in Φ is true in A_0. Now extend A_0 to an λ-saturated model, A. We now play a game between A and B to get $A' \preccurlyeq A$ and $B' \preccurlyeq B$ with an embedding from A' into B', such that B' end-extends the image of A'. We first show that $\exists$ can always win the usual game. Suppose, for $i < \lambda$, that $\bar{a}(i)$ from A, $\bar{b}(i)$ from B, and C_i a subset of B have been chosen already, with $(A, \bar{a}(i)) \Rrightarrow_{\Sigma_1^0} (B, \bar{b}(i))$. Let $\forall$ choose $a_i \in A$. Let p be the Σ_1^0 type of a_i over $\bar{a}(i)$. It is clear that p is finitely satisfied over $\bar{b}(i)$, and thus that it is satisfied in B, by b, so $\bar{a}(i+1) = \bar{a}(i) \frown a_i$ and $\bar{b}(i+1) = \bar{b}(i) \frown b$. $C_{i+1} = C_i$. Now suppose $\forall$ chooses b_i in B. Let $\Theta(\bar{y}) = \{\neg\theta(x, \bar{y}) \mid B \models \theta(b_i, \bar{b}(i)) \wedge \theta \in \Sigma_1^0\}$. If $\Theta(\bar{a}(i))$ is finitely satisfiable in A, then it is satisfiable, so let a be a realization, and let $\bar{a}(i+1) = \bar{a}(i) \frown a$ and $\bar{b}(i+1) = \bar{b}(i) \frown b_i$, $C_{i+1} = C_i$. Suppose $\Theta(\bar{a}(i))$ is not finitely satisfiable in A. Then we can find θ such that $A \models \forall x \theta(x, \bar{a}(i))$, where θ is a Σ_1^0 formula whose negation is in Θ. If $\neg b_i < b$ for every $b \in \bar{b}(i)$, let $C_{i+1} = C_i \cup \{b_i\}$, $\bar{a}(i+1) = \bar{a}(i)$, $\bar{b}(i+1) = \bar{b}(i)$. Now assume $b_i < b$, for some $b \in \bar{b}(i)$. Then B satisfies the formula $\exists x < b(\theta(x, \bar{b}(i)))$. But then, since this formula is Σ_1^0, so does A, which is a contradiction. Thus, $\exists$ can always extend the map or append an element to C. The game can thus be played to λ-many steps, yielding $A' = \bigcup_{i<\lambda} \bar{a}(i)$ and $B' = \bigcup_{i<\lambda} \bar{b}(i) \cup C_i$.

We now give instructions to $\forall$. For convenience, let there be three $\forall$ players, $\forall_A$, $\forall_B$, and $\forall_C$. Each is assigned a cofinal sequence in λ. If it is $\forall_B$'s turn at $i+1$, he considers all formulas such that $A \models \exists x \varphi(x, \bar{b})$, for $\bar{b} \subseteq \bar{b}(i) \cup C_i$, consistently well-ordering them. For the first one in his list, he chooses a witness from B, b_i. Likewise for $\forall_A$ (without the C_i). $\forall_C$'s job is different. He keeps track of the elements in C, and maintains a well-ordered list of elements c such that for some $b \in \bar{b}(i)$, $c < b$, but $c \notin \bar{b}(i)$. Every time a choice is made that gives more elements in C this property, he adds them on to the end of the list. Since $|C_i| \leq i$, his list always has length $< \lambda$. When it is his turn, he chooses the first element on the list, which permanently removes it since a match is found. Every element on his list has index $< \lambda$, and he has λ many opportunities, so each will be taken care of. This procedure results in an end-embedding of A' into B', with A' and B' elementary embeddings of A and B respectively.

Thus, ϕ is true in B', thus in B, and thus in B_0. Therefore, by compactness, some positive boolean combination of the Σ_1^0 consequences of ϕ imply it, and so ϕ is equivalent to a Σ_1^0 formula.

(b) implies (a) is Exercise 2.4.5.

8.3.7*. Let L be a first-order language whose symbols include a 2-ary relation symbol $<$, and let T be a theory in L which implies '$<$ linearly orders the set of all elements, with no last element.' Recall from Exercise 2.4.9 the notion of a *cofinal substructure*; let Φ be defined as in that exercise, except that the formulas in Φ are required to be first-order. Show (a) if A and B are models of T, A and B are $|L|$-saturated and $A \Rrightarrow_\Phi B$, then there is an embedding of an elementary substructure A' of A onto a cofinal substructure of an elementary substructure B' of B, (b) if ϕ is a formula of L, and f preserves ϕ whenever f is an embedding of a model of T onto a cofinal substructure of a model of T, then ϕ is equivalent modulo T to a formula in Φ.

(a) First, note that the statement is incorrect. To see this, let $L = \{<,\}$, and let A be $\mathbb{Q}$ with $P(A)$ dense in A, and B be $\mathbb{Q}$ with $B \models \forall x P(x)$. Then both A and B are $|L|$-saturated, and both admit elimination of quantifiers. Thus, $A \Rrightarrow_\Phi B$. However, any elementary substructure of A includes elements such that $\neg P(x)$ is true, and thus those elements cannot map to elements in B. The reversed statement, however, is true: if $B \Rrightarrow_\Phi A$, then the conclusion follows. This is what I prove. Just for fun, I also make B and A $|L|^+$-saturated.

We have $\exists$ and $\forall$ play a game similar to the one above. After stage i, we have $\bar{a}(i)$, $\bar{b}(i)$, and C_i as above, with $(B, \bar{b}(i)) \Rrightarrow_\Phi (A, \bar{a}(i))$. Let $\forall$ choose a_i. Then let $\Theta(x, \bar{y}) = \{\neg\theta(x, \bar{y}) \mid A \models \theta(a, \bar{a}(i)) \wedge \theta \in \Phi\}$. If $\Theta(x, \bar{b}(i))$ is finitely satisfiable, it is satisfiable, so we can choose b satisfying it. If not, then we have $B \models \forall x \bigvee_{j<n} \theta_i(x, \bar{b}(i))$. But it is easy to see that this sentence is in Φ, and thus true in $(A, \bar{a}(i))$, which is impossible. Now let $\forall$ choose b_i. Then let $\Theta(x, \bar{y}) = \{\theta(x, \bar{y}) \mid A \models \theta(a, \bar{a}(i)) \wedge \theta \in \Phi\}$. If Θ is finitely satisfiable, append the realization, as usual. Otherwise, just append it to C_i.

Now, we specify what $\forall$ is doing. As before, there are three $\forall$s. The first two are doing the same as before. The third, $\forall_C$, has the following task. He keeps a well-ordered list of C_i. When it is his turn, he takes the first c such that $c > b$ for every $b \in \bar{b}(i)$. Now he tries to find a such that a can be mapped to an element $b > c$. He does this as follows. Choose a random a. By the above argument, we can certainly find a b such that $(B, \bar{b}(i), b) \Rrightarrow_\Phi (A, \bar{a}, a)$. However, suppose there is no such b for $b > c$. By compactness then, we can write $B \models \forall x > c(\theta(x, \bar{b}(i)))$. Thus, $B \models \exists z \forall x(z < x \rightarrow \theta(x, \bar{b}(i)))$. This means that A satisfies the same sentence. Let a_1 be the witness for z. Now repeat with some choice $a > a_1$. After λ-many steps, we will either have found an a, or run out of formulas, thus finding an a. Now let $\forall_C$ choose the corresponding b. Then $\exists$ will find an a to map b to, and we will no longer have $b < c$ for $b \in \bar{b}(i)$.

(b) Let A be an $|L|^+$-saturated model of the Φ-consequences of ϕ. We can find B, an $|L|^+$-model of ϕ, with no new formulas of Φ true in it. Then the above procedure shows that ϕ is true in A.

8.3.8. Let L be a first-order language whose symbols include a 2-ary relation symbol R, and let T be a theory in L. Suppose that T implies that R expresses a reflexive symmetric relation. If A is a model of T, we define a relation $\sim$ on $\mathrm{dom}(A)$ by '$\sim$ is the smallest equivalence relation containing R^A. A closed substructure of A is a substructure whose domain is a union of equivalence classes of $\sim$. We define Θ to be the least set of formulas of L such that (i) Θ contains all quantifier-free formulas, (ii) Θ is closed under disjunction, conjunction and existential quantification, and (iii) if $\phi(\bar{x}yz)$ is in Θ, and y occurs free in ϕ, then the formula $\forall z(Ryz \to \phi)$ is in Θ. Show (a) if A and B are models of T, A and B are $|L|$-saturated and $A \Rrightarrow_\Theta B$, then there is an embedding of an elementary substructure A' of A onto a closed substructure of an elementary substructure B' of B, (b) if ϕ is a formula of L, and f preserves ϕ whenever f is an embedding of a model of T onto a closed substructure of a model of T, then ϕ is equivalent modulo T to a formula in Θ.

(a) The argument is much the same as in 8.3.6. The only difference for $\exists$ is when $\forall$ chooses an element of B, b_i. As before, if $\Phi(x, \bar{y}) = \{\neg\theta(x, \bar{y}) \mid \theta \in \Theta \wedge B \models \neg\theta(b_i, \bar{b}(i))\}$ is finitely satisfiable in A over $\bar{a}(i)$, then there is no problem. Otherwise, if for every $b \in \bar{b}(i)$, $\neg Rbb_i$, then add b_i to C_i. Finally, if Rb_kb_i for some $b_k \in \bar{b}(i)$, then write $A \models \forall x\theta'(x, \bar{a}(i))$ with $B \models \neg\theta'(b_i, \bar{b}(i))$ (since Θ is closed under disjunctions), and then $A \models \forall x(Ra_kx \to \theta'(x, \bar{a}(i)))$, but since $(A, \bar{a}(i)) \Rrightarrow_\Theta (B, \bar{b}(i))$, B satisfies the corresponding sentence, which is impossible.

$\forall_C$ is the only player whose procedure changes. He keeps a consistent well-ordered list of elements of C_i such that for some $b \in \bar{b}(i)$, Rcb but $c \notin \bar{b}(i)$. When it is his turn, he chooses the first such c. The end result will be as desired.

(b) The argument is the same as 8.3.6.

8.3.9. Let ϕ be the sentence $\exists xy(Rxy \wedge \forall z(Rxz \to \exists t(Rxt \wedge Rzt)))$. Show (a) if a structure A is a homomorphic image of a model of ϕ, then A contains elements a_i ($i < \omega$), not necessarily distinct, such that $A \models R(a_0, a_i) \wedge R(a_i, a_{i+1})$ whenever $0 < i < \omega$, (b) if a structure B contains arbitrarily long finite sequences like the sequence of length ω in (a), then some elementary extension of B is a homomorphic image of a model of ϕ, (c) by (a) and (b), the class of homomorphic images of models of a first-order sentence need not be closed under elementary equivalence.

(a) Work in the model C with A a homomorphic image of C. Find a witness for x in ϕ, c_0, and for y, c_1. Given $(c_0, \ldots, c_i)$, find c_{i+1} by setting z to c_i in ϕ, and letting c_{i+1} be a witness. Then we can continue this to a sequence $\langle c_i \mid i < \omega\rangle$. The image of this sequence in A is then the desired one.

(b) By compactness, extend B to an elementary extension B' with a sequence of length ω. Let $\langle b_i \mid i < \omega\rangle$ be the sequence in B'. Let C be the same model as B', except that for any $c_1, c_2 \in C$, Rc_1c_2 if and only if either $c_1 = b_0$ and $c_2 = b_i$ for some $i \in \omega$, or $c_1 = b_i$ and $c_2 = b_{i+1}$ for some $i \in \omega$. Then $C \models \phi$, and B' is a homomorphic image of C.

(c) Let B be ω^*, and R the usual ordering. Then B has no infinite ascending sequences, and so is not the homomorphic image of a model of ϕ, but some elementary extension is, so elementary equivalence does not preserve the property.

8.4.1. Let L be a countable first-order language, T a theory in L and $\phi(x)$, $\psi(x)$ formulas of L. Suppose that for every model A of T, $|\psi(A)| \leq |\phi(A)| + \omega$. Show that there is a polynomial $p(x)$ with integer coefficients, such that for every model A of T, if $|\phi(A)| = m < \omega$ then $|\psi(A)| \leq p(m)$.

By Vaught's two-cardinal theorem, $\psi(x) \leq \phi(x)$, since if not, we can find a model, A, where $|\psi(A)| = \omega_1$ and $|\phi(A)| \leq \omega$. Thus, we can layer ψ by ϕ. Let the layering be $\theta(x)$, in the form given in equation (4.3). Let A be a model of T in which $\phi(A)$ is finite.

For each $a \in \psi(A)$, for some $k < n$, $\theta(a)$ is satisfied with $a = z_k$. Make n sets $P_k \subseteq \psi(A)$, containing all a's of this form. We make m subsets of each P_k as follows. For each $b \in \phi(A)$, let $P_k(b)$ be the subset of P_k such that θ's satisfaction uses b for y_0. There are thus m subsets. Restrict to one of these. Now, if $\exists z_0 \eta(y_0, z_0)$, then there is a unique z_0, say c_0. Otherwise, choose any c_0. Then subdivide further into m sets based on what value y_1 takes. Repeating, in the end we have $\eta_k(b_0, c_0, \ldots, b_k, z_k)$, with only one solution for z_k. Thus, after dividing into m subsets $k+1$ times, each subset can have at most one element. Thus, $|P_k| \leq m^{k+1}$, so $|\psi(A)| \leq \sum_{i<n} m^{i+1}$.

8.4.2. Let L be a first-order language, T a complete theory in L with infinite models, and $\phi(x)$, $\psi(x)$ formulas of L. By a stratification of ψ over ϕ in a model A of T we mean a formula $\sigma(x, y)$ of L with parameters from A, such that $A \models \forall x(\psi(x) \leftrightarrow \exists y(\sigma(x, y) \wedge \phi(y)))$; we call the stratification σ algebraic if for every element b of $\phi(A)$, the set $\{a \mid A \models \sigma(a, b)\}$ is finite. Show (a) even when $\psi \leq \phi$, there need not be an algebraic stratification of ψ over ϕ in any model of T, (b) if A is a model of T, $\psi \leq \phi$ and $\psi(A)$ is infinite, then there are a formula $\rho(x)$ of L with parameters from A such that $\rho(A)$ is an infinite subset of $\psi(A)$, and an algebraic stratification of ρ over ϕ in A.

(a) Let T be the theory of an infinite set and pairs of elements of the set. We have 2 "sorts," picked out by unary F and G, and relations $comp_0$, and $comp_1$ on $G \times F$ picking out each component. Let $\phi(x)$ be $F(x)$, and let $\psi(x)$ be $G(x)$. Then clearly $|\phi(A)| = |\psi(A)|$ for every $A \models T$. Moreover, if $\rho(x, y)$ is any formula, it is easy to translate ρ into a formula in the empty language, and then eliminate quantifiers. Thus, $\rho(x, y)$ can be rewritten as $\rho'(x_0, x_1, y)$, quantifier-free, in the empty language. Now suppose that for every b, $\rho'(x_0, x_1, b)$ is finite. Then $\exists x_1 \rho'(x_0, x_1, b)$ is finite. Thus, by eliminating quantifiers again, any x_0 satisfying this formula must be b. Likewise for x_1. Thus, (b, b) is the only element satisfying $\rho'(x_0, x_1, b)$. Thus, there are some (a, b) disproving that ρ is a stratification.

(b) Let $\theta(x)$ be a layering of ψ by ϕ, with the usual form for θ. Write $\theta_k(x, y_0, z_0, y_1, \ldots, z_{k-1}, y_k)$ for the formula which results if we delete everything up to $(\exists y_k \in \phi)$ inclusive. There are now n cases. Case i is: i is the least number such that for each $b \in \phi(A)$ there are only finitely many $a \in \psi(A)$ such that $A \models \theta_i(a, b_0, c_0, b_1, \ldots, b)$, where for $j < i$ $\{a \mid A \models \theta_j(a, b_0, c_0, b_1, \ldots, b_j)\}$ is infinite, and c_j for $j < i$ is always either the unique element such that $A \models \eta(b_0, c_0, \ldots, b_j, c_j)$ if there is such an element, or some arbitrary element otherwise. Then put $\rho = \theta_{i-1}(x, b_0, c_0, \ldots, b_{i-1})$, and $\sigma(x, y) = \theta_i(x, b_0, c_0, \ldots, y)$ (if

$i = 0$, put $\rho = \psi$). We verify that these choices work. By the first sentence, there are only finitely many solutions to $\sigma(x, b)$ for any b. By the second sentence, ρ is infinite. By the layering, if $\rho(x)$, then $\theta(x)$, so $\psi(x)$, and by the definitions, $A \models \forall x(\rho(x) \leftrightarrow \exists y(\sigma(x, y) \wedge \phi(y)))$. It remains to be shown that one of these cases is true. But as we consider each $\theta_j(x, b_0, c_0, \ldots, -)$, either there is or isn't a b which gives infinitely many solutions. If there isn't, we are done here, and if not, set $b_j = b$ and move on. Suppose we have done this for every case. Then we have $b_0, c_0, \ldots, b_{n-1}$ such that $\theta(x, b_0, c_0, \ldots, b_{n-1})$ has infinitely many solutions in ψ. Then, choosing c_{n-1} according to the usual procedure, we have a contradiction, since x must be equal to some c_i.

9.1.1. Give an example to show that if F is an EM functor and $f : \omega \to \omega$ an order-preserving map, $F(f) : F(\omega) \to F(\omega)$ need not preserve $\forall_1$ first-order formulas.

Let F be the identity map, so $F(\eta) = (\eta, <^\eta)$. Let $f : \omega \to \omega$ be the map $f(i) = i + 1$. But the map $i \to i + 1$ does not preserve the formula $\forall y(y = x \vee y > x)$, since 0 maps to 1.

9.1.2. Show that if T is any first-order theory with infinite models and G is a group of automorphisms of a linear ordering η, then there is a model A of T which contains η, such that G is the restriction to η of a subgroup of $\mathrm{Aut}(A)$. In particular show that T has a model on which the automorphism group of the ordering $(\mathbb{Q}, <)$ of the rationals acts faithfully.

We can find an EM functor to models of T, F. Consider $A = F(\eta)$. Let f be any automorphism of η. Then $F(f)$ extends to an embedding of A into A, which is onto since f is onto, and is thus an automorphism. Thus, $F(\mathrm{Aut}(\eta)) \subseteq \mathrm{Aut}(A)$.

9.1.3. Let F be an EM functor in the first-order language L, with skolemised theory. Show that if η is any infinite linear ordering and X is any set which is first-order definable in $F(\eta)$ without parameters, then X has cardinality either $|\eta|$ or $\leq |L|$.

Let $\varphi(x)$ define X. Since we have a skolem theory, $\varphi(x)$ is quantifier-free. Consider any $a \in X$. Write $a = t(\bar{c})$ for some increasing tuple $\bar{c}$ in η. Then $\varphi(t(\bar{x})) \in \mathrm{Th}(F)$. Thus, $t(\bar{d}) \in X$ for any increasing tuple $\bar{d}$ in η. The question is now whether $t(\bar{x}) = t(\bar{y})$ for all increasing tuples $\bar{x}$, $\bar{y}$ in η. However, there are several possible orderings for $\bar{x}$ and $\bar{y}$ when considered as a single increasing tuple $(x_1 < x_2 < y_1 < x_3 < y_2 < \ldots)$. On the other hand, there are only finitely many. If for any such ordering we have $t(\bar{x}) \neq t(\bar{y}) \in \mathrm{Th}(F)$, then clearly $|X| = |\eta|$. If not, then this term adds a unique element to X. Thus, if there is any $a \in X$ such that its terms are all unique, then $|X| = |\eta|$, and if every a is the result of a unique term, then X is bounded by the number of terms, or $|L|$.

9.1.4. Let L and L^+ be first-order languages with $L \subseteq L^+$ and suppose every symbol in L^+ but not in L is a relation symbol. Let F be an EM functor in L and T an $\forall_1$ theory in L^+ which is consistent with $\mathrm{Th}(F(\omega))$. Show that there is an EM functor F^+ in L^+ such that for each linear ordering η, $F^+(\eta)$ is a model of T and $F(\eta) = F^+(\eta)|L$.

We show that there is a way to extend F to the required F^+. By Theorem 9.1.4, we need only define $F^+(\omega)$. Let $\lambda = |L^+ \setminus L|$. Enumerate the relations $\langle R_i \mid i < \lambda\rangle$. Let $L_i = L \cup \{R_j \mid j < i\}$. We construct a sequence of models, $\langle A_i \mid i < \lambda\rangle$, such that A_i is an L_i-structure containing ω as a sequence of generators which is indiscernible for atomic formulas of L_i, $A_i|L_j = A_j$, and $\mathrm{Th}(A_i)$ is consistent with T. Set $A_0 = F(\omega)$. Proceed by induction. At limit stages, take "unions." We are left with the successor case, $k = i + 1$. Let $A = A_k$. Let $R = R_k$. Let n be the arity of R. All we need to specify about A is the following: for every $t_1(\bar{x}_1), \ldots, t_n(\bar{x}_n)$ terms of L^+ (also terms of L), we must decide whether $\theta = R(t_1(\bar{x}_1), \ldots, t_n(\bar{x}_n))$ is true for every increasing tuple of ω, since knowing all of this completely determines R^A. (Note that θ is dependent on the ordering of the $\bar{x}_j$'s, so there are finitely many versions for each choice of the t_j's). Let $\langle \theta_j \mid j < |L|\rangle$ be an enumeration of these θ's. We define $\langle U_j \mid j < |L|\rangle$, a chain of sets of sentences of $L_k(A_i)$. Then $U = \bigcup_{j<|L|} U_j$ will specify R^A. We will ensure U is consistent by making each U_j consistent with T, will make sure that each U_j preserves the indiscernibility of ω, and will also prove that $\mathrm{Th}(A)$ is consistent with T. $U_0 = \mathrm{diag}(A_i)$. Then U_0 is consistent with T: if not, then $T \vdash \neg\varphi(\bar{a})$, where $\varphi(\bar{a}) \in \mathrm{diag}(A_i)$. But then $T \vdash \forall\bar{x}\neg\varphi(\bar{x})$, while $\mathrm{Th}(A_i) \vdash \exists\bar{x}\varphi(\bar{x})$, which is impossible, since $\mathrm{Th}(A_i)$ and T are consistent. Limits are unions, so all that remains is the successor case. If $T \vdash \theta_j$ (or if $T \vdash \neg\theta_j$), then define U_{j+1} to be U_j together with $\theta_j(\bar{a})$ (or $\neg\theta_j(\bar{a})$) for every $\bar{a}$ an ascending tuple in ω. We show U_{j+1} is consistent. Assume not. Then (say) $U_j \vdash \theta_j(\bar{a})$. But by assumption, U_j is consistent with T, and $T \vdash \forall\bar{x}\neg\theta_j(\bar{x})$, so this is impossible. The other case is handled the same way. Now, assume T does not decide θ_j. If there is some sentence in U_j of the form $\theta_j(\bar{a})$ (or $\neg\theta_j(\bar{a})$) for some $\bar{a}$ in ω, then by indiscernibility, U_j decides θ_j, so U_{j+1} is made in the obvious way. If none of the above cases occur, let U_{j+1} be U_j together with $\theta_j(\bar{a})$ for every $\bar{a}$ an ascending tuple in ω. Suppose U_{j+1} is not consistent with T. Then $T \vdash \neg\theta_j(\bar{a}_1) \vee \cdots \vee \neg\theta_j(\bar{a}_m)$, for some $\bar{a}_1, \ldots, \bar{a}_m$ increasing tuples in ω. By the lemma on constants, $T \vdash \forall\bar{x}_1, \ldots, \bar{x}_m(\neg\theta_j(\bar{x}_1) \vee \cdots \vee \neg\theta_j(\bar{x}_m))$, but this clearly implies $T \vdash \forall\bar{x}\neg\theta_j(\bar{x})$, so T did decide θ.

Now, let $U = \bigcup_{j<\lambda} U_j$. U specifies R^A, so we have now defined A. We must verify that $\mathrm{Th}(A)$ is consistent with T. Suppose not. Since T is $\forall_1$, we have $\mathrm{Th}(A) \vdash \exists\bar{x}\theta(\bar{x})$, for some θ quantifier-free, with $T \vdash \forall\bar{x}\neg\theta(\bar{x})$. Let $(t_1(\bar{a}), \ldots, t_m(\bar{a}))$ be witnesses in A, with $\bar{a}$ some increasing tuple in ω. Then $\mathrm{Th}(A) \vdash \theta((t_1(\bar{a}), \ldots, t_m(\bar{a})))$. Writing θ as a disjunction of conjunctions, we can further reduce to the case that θ is a conjunction (since these terms satisfy one specific conjunction in the disjunctions). But then θ is a conjunction of sentences in U, and U is consistent with T, contradiction.

Let $B = \bigcup_{i<\lambda} A_i$. Then $B|L = F(\omega)$, $B \models T$, and ω is a sequence of generators which is indiscernible for atomic sentences. Then there is a unique F^+ such that $F^+(\omega) = B$. Since T is $\forall_1$, for any η, $F^+(\eta)$ will be a model of T. For every η, $F^+(\eta)|L = F(\eta)$, since if not, we can use a sliding argument to translate the discrepancy into $F(\omega)$ and $F^+(\omega)|L$, which are the same.

9.1.5*. Let L be a first-order language containing a 2-ary relation symbol $<$, and A an L-structure such that $<^A$ linearly orders the elements of A in order-type κ for some infinite cardinal κ. Writing η for the ordering $(\operatorname{dom} A, <^A)$, show that $\operatorname{Th}(A, \eta)$ contains the following formulas: (i) '$<$ linearly orders the universe' and $x_0 < x_1$; (ii) for each term $t(x_0, \ldots, x_{n-1})$ of L, the formula $t(x_0, \ldots, x_{n-1}) < x_n$; (iii) for each term $t(x_0, \ldots, x_{n-1})$ of L and each $i < n$, the formula $t(x_0, \ldots, x_{n-1}) \leq x_i \to t(x_0, \ldots, x_{n-1}) = t(x_0, \ldots, x_i, x_n, x_{n+1}, \ldots, x_{2n-(i+2)})$.

'$<$ linearly orders the universe' is $\forall xy((x < y \vee y < x \vee y = x) \wedge (x < y \to (\neg y < x \wedge \neg y = x) \wedge x = y \to \neg x < y))$. Since it is true in A, it is in $\operatorname{Th}(A, \eta)$. $x_0 < x_1$ is in $\operatorname{Th}(A, \eta)$ since every increasing pair is just that.

The second statement is false, as seen by considering $(\omega, S, <)$, where S is the successor function, and $Sx_0 < x_1$. When $x_1 = Sx_0$, it is not true. The third statement is also incorrect: to contradict the original, consider $(\omega, +, -, <)$, where $-$ is the modified $-$ function. Then consider $x_0 + x_1 + x_2 - x_3$. For the choice $(1, 2, 3, 5)$, we have $x_0 + x_1 + x_2 - x_3 \leq x_0$. But $1 \neq 1 + x_4 + x_5 - x_6$ for every choice of $x_4, x_5, x_6 \in \omega$, in particular, $(1, 2, 3, 5, 6, 7, 8)$ contradicts this formula.

Modify the problem by adding the condition η is an indiscernible sequence with respect to atomic formulas. Now consider $t(x_0, \ldots, x_{n-1})$. For a given $a_0, \ldots, a_{n-1}$ an increasing tuple in A, $t(a_0, \ldots, a_{n-1})$ is some element. Since A has order type κ, there is some element of A above $\max(t(a_0, \ldots, a_{n-1}), a_0, \ldots, a_{n-1})$. Choosing this element for a_n, we have $t(a_0, \ldots, a_{n-1}) < a_n$, and thus by indiscernibility, condition (ii).

For (iii), find $a'_0, \ldots, a'_{n-1}$ such that the if-clause holds, otherwise the formula is trivially in $\operatorname{Th}(A, \eta)$. Set $b = t(a'_0, \ldots, a'_{n-1})$. Assume $b \neq a_j$ for any $j \leq i$. Rename the a'_is and b so that we have $a_0, \ldots, a_n$, with $b = a_k$ for some $k \leq i$. Choose arbitrary $a_{n+1}, \ldots, a_{2n-i+3}$. By indiscernibility, $t(a_0, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{i+1}, a_{n+1}, \ldots, a_{2n-i+3}) = a_k$, so we are done. If $b = a_k$, for some $k \leq i$, then no renaming is necessary, as $t(a'_0, \ldots, a'_i, a_n, \ldots, a_{2n-i-2)}) = a'_k$ by indiscernibility.

9.1.6*. Let L be a first-order language containing a 2-ary relation symbol $<$ and F an EM functor for L which contains all the formulas (i)-(iii) of the preceding exercise. Show that if λ is any infinite cardinal, then in $F(\lambda)$ there are no $<$-descending sequences of length $|L|^+$, the spine is cofinal and no element α of the spine has more than $|\alpha| + |L|$ predecessors in the $<$-ordering.

The first and third claims are incorrect as given. Let $F(\omega_1)$ be $(\omega_1^* + \omega_1, <, f)$, where ω_1^* is the reverse of ω_1, and f is a unary function taking $i \in \omega_1$ to $i \in \omega_1^*$ and vice versa. Then ω_1 is a set of generators of $F(\omega_1)$, and is indiscernible with respect to atomic formulas, but there is a descending sequence of length $\omega_1 > |L|$. This model also gives a counterexample to the third claim. To fix this, we could add the following formula to (iii): $t(x_0, \ldots, x_{n-1}) < x_i \to t(x_0, \ldots, x_{n-1}) = t(x_0, \ldots, x_{i-1}, x_n, \ldots, x_{2n-i-1})$. It is in (A, η) by the same arguments. However, this does not fix the problem with the first statement: consider the structure with universe $\omega_2 \cup \{(\alpha, \beta) \mid \beta < \alpha \in \omega_2\}$, ordered by the usual ordering on ω_2, the lexicographic ordering with the second coordinate reversed on (α, β) (so $(\alpha, \beta) < (\alpha + 1, \gamma)$, and $(\alpha, \beta) > (\alpha, \beta + 1)$), and $\alpha < (\alpha, \beta) < \alpha + 1$. As well, let $f(\alpha, \beta) = (\alpha, \beta)$ with $\beta < \alpha \in \omega_2$, and $f(a, b) = a$ otherwise. This ordering consists of ω_2 with a reversed copy of each ordinal inserted between it and its successor. ω_2 is a generating set, and an indiscernible sequence with respect to atomic formulas, but there is a descending sequence of length $\aleph_1$.

The second statement is trivial, since $t(x_0, \ldots, x_{n-1}) < x_n$.

The third statement, with the modified (iii), goes as follows. Let X be the set of all predecessors of α. By (iii), for any $b \in X$, if $b = t(\bar{a})$, with every element in $\bar{a} \geq \alpha$, then t is a constant term on λ, so there are at most $|L|$ such elements in X. There are at most $\alpha^{<\omega} \leq \alpha + |L|$ elements in X of the form $t(\bar{a})$ with every element in $\bar{a} \leq \alpha$. Thus, this leaves only elements of the form $t(a_0, \ldots, a_{n-1})$ with $a_0, \ldots, a_{i-1} \leq \alpha$, and $a_i, \ldots, a_{n-1} > \alpha$. But then by (iii), $a_0, \ldots, a_{i-1}$ determine the value of t, and there are only $\alpha^i \leq \alpha + |L|$ possibilities.

9.2.1. Show that a minimal set remains minimal if we add parameters; likewise with 'strongly minimal' for 'minimal'.

Since the formulas we can use are allowed to have parameters from the model, adding parameters to the language changes nothing.

9.2.2. Give an example to show that the notion 'algebraic over' need not obey the exchange law.

Let L be the language with one unary function symbol, f, and let $A = \omega \oplus \omega$, with $f(i,j) = (i,0)$. Then $\mathrm{acl}(\emptyset) = \emptyset$, $(1,0) \in \mathrm{acl}((1,1))$, but $(1,1) \notin \mathrm{acl}((1,0))$.

9.2.3. Let G be a group. We define a *G-set* to be an L-structure A as follows. The signature is a family of 1-ary functions $(F_g \mid g \in G)$; the laws $\forall x F_g F_h(x) = F_{gh}(x)$, $\forall x F_1(x) = x$ hold in A. We say A is a *faithful* G-set if for all $g \neq h$ in G and all elements a, $F_g^A(a) \neq F_h^A(a)$. (a) Show that the class of faithful G-sets is first-order axiomatisable, in fact by $\forall_1$ sentences. (b) Show that if A is an infinite faithful G-set, then A decomposes in a natural way into a set of connected components and each component is isomorphic to a Cayley graph of the group G (see Exercise 4.1.1 above). (c) Deduce that every infinite faithful G-set is strongly minimal, and its dimension is the number of components.

(a) The axioms given above are easily seen to be $\forall_1$.

(b) We first show that if A is a faithful G-set, then the above axioms, denoted by T, are model-complete, by Lindström's test. We must show that T is λ-categorical for some $\lambda \geq |L|$. We construct an isomorphism between any two infinite models $M \models T$ and $N \models T$ with $|M| = |N| > |G|$. Enumerate M as $(a_0, a_1, \ldots)$, and N as $(b_0, b_1, \ldots)$. To start, we have $\mathfrak{M} \equiv_0 \mathfrak{N}$, since there are no quantifier-free sentences. Map a_0 to b_0. This induces a mapping of $|G|$ elements of M to $|G|$ elements of N, which will respect the F_g's, so we will have $(\mathfrak{M}, a_0, \ldots) \equiv_0 (\mathfrak{N}, b_0, \ldots)$. Since $|N| > |G|$, there is a first element of N not yet mapped to, b_i. Map it to the first element of M not yet mapped to. Continue this process. We get an isomorphism. Thus, by Lindström's test, T is model-complete. Moreover, it is easy to see that models of $T_\forall (= T)$ have the amalgamation property. Thus, by Theorem 7.4.1, T has quantifier elimination. This means that if b is algebraic over a, it is in fact equal to $F_g(a)$, for some $g \in G$. Thus, we can break A into its connected components, $\{X \mid X = \mathrm{acl}(a), a \in A\}$. Each X looks like a copy of G with the F_g's acting on it, and so is precisely the Cayley graph.

(c) By quantifier elimination, any formula clearly defines a finite or cofinite set, so a faithful infinite G-set is strongly minimal. Since a set with one element from each component has algebraic closure the entire structure, and no smaller set can possibly work, the dimension is the number of components.

9.2.4*. Let A be an L-structure, Ω a $\emptyset$-definable minimal set in A, $(a_i \mid i < \lambda)$ a sequence of elements of Ω, X a set of elements of A and $\bar{b}$ a sequence listing the elements of X. Show that the following are equivalent: (a) $(a_i \mid i < \gamma)$ is independent over X; (b) for each $i < \gamma$, $a_i \notin \operatorname{acl}(X \cup \{a_j \mid j < i\})$; (c) $a_i \neq a_j$ whenever $i < j$, and $\{a_i \mid i < \gamma\}$ is an independent set in $(A, \bar{b})$.

I see no difference between (a) and (c). We show (a) is equivalent to (b). It is clear that (a) implies (b). Assume (a) is false. Then let $a_i \in \operatorname{acl}(X \cup \{a_j \mid j \neq i\})$. Let $\varphi(x, \bar{a})$ (with other parameters in X) be a formula witnessing this with $\bar{a}$ a tuple of a_j's, and minimal with respect to the length of $\bar{a}$ among all such formulas. Let k be the maximum index appearing in $\bar{j}$. If $k < i$, then (b) is false and we are done. Moreover, $a_i \notin \operatorname{acl}(X \cup \bar{a} \setminus \{a_k\})$, because if so, that would yield a formula with a shorter $\bar{a}$. Thus, for $Y = X \cup \bar{a} \setminus \{a_k\}$, we have $a_i \in \operatorname{acl}(Y \cup \{a_k\}) \setminus \operatorname{acl}(Y)$, so by exchange, $a_k \in \operatorname{acl}(Y \cup \{a_i\})$, showing that (b) is false.

9.2.5. Suppose $\phi(x)$ is a formula of L and $\phi(A)$ is a minimal set in A with dimension κ. If $|L| \leq \lambda \leq \kappa$, show that A has an elementary substructure B in which $\phi(B)$ has dimension λ.

Let X be a basis of $\phi(A)$, with $|X| = \kappa$, and let Y be a subset of X with $|Y| = \lambda$. We will construct a B along the lines of Löwenheim-Skolem, but taking some care. Let $B_0 = \langle Y \rangle_A$. We build a chain of structures of length λ, $(B_i \mid i < \lambda)$, such that for each $\varphi(x, \bar{b})$, with $\bar{b}$ in B_i, if $A \models \exists x \varphi(x, \bar{b})$, then there is $c \in B_{i+1}$ with $B_{i+1} \models \varphi(c, \bar{b})$. This is done as follows. (Note that $\langle C \rangle_A \subseteq \operatorname{acl}_A(C)$, for any $C \subseteq A$.) Let $\langle \theta_i \mid i < \lambda \rangle$ enumerate formulas of the form $\exists x \varphi(x, \bar{b})$, with $\bar{b}$ in B_i. We form a chain $\left\langle B_i^{(j)} \mid j < \lambda \right\rangle$. $B_i^{(0)} = B_i$. If $A \models \neg\theta_j$, $B_i^{(j+1)} = B_i^{(j)}$. If $\theta_j = \exists x \varphi(x, \bar{b})$ and there exists $d \in B_i^{(j)}$ with $A \models \varphi(d, \bar{b})$, $B_i^{(j+1)} = B_i^{(j)}$. If there is no such $d \in B_i^{(j)}$, choose a witness d from A and set $B_i^{(j+1)} = B_i^{(j)} \cup \{d\}$. Let $B_{i+1} = \bigcup_{j<\lambda} B_i^{(j)}$. Note that B_{i+1} is a model, because for any constant c, the formula $\exists x(x = c)$ has a solution in A, and for any function f and any $\bar{b}$, $\exists x(f(\bar{b}) = x)$ has a solution in A. Let $B = \bigcup_{i<\lambda} B_i$. Certainly $\dim(\phi(B)) \geq \lambda$, so we must prove that Y actually spans $\phi(B)$. Consider any $b \in \phi(B)$. If $b \in Y$, certainly $b \in \operatorname{acl}_B(Y)$, so assume not. Moreover, let b be the first such element inserted into B by the above process. By the construction of B, there is some formula $\psi(x, \bar{a})$, with $A \models \psi(b, \bar{a})$, for some $\bar{a}$ in $\operatorname{acl}(Y)$, which is responsible for b being in B. Let $\theta(x, \bar{a}) = \psi(x, \bar{a}) \wedge \phi(x)$. If θ has finitely many solutions in A, then b is algebraic over $\bar{a}$, and so we are done (since $\operatorname{acl}(\operatorname{acl}(Y)) = \operatorname{acl}(Y)$). Suppose θ has infinitely many solutions in A, and therefore in $\phi(A)$. Then it has cofinitely many solutions in $\phi(A)$, and hence some $c \in Y$ satisfies $\theta(c, \bar{a})$, since $|Y| \geq |L| \geq \omega$. Thus, $\psi(x, \bar{a})$ cannot have been responsible for b's entrance, contradiction. Thus, Y does span $\phi(B)$, so $\dim(\phi(B)) = \kappa$.

9.2.6*. Show that if $\psi(A)$ is a minimal set in A, then $\psi(A)$ is strongly minimal if and only if A has an elementary extension B in which $\psi(B)$ is minimal and of infinite dimension. In particular, every minimal structure of infinite dimension is strongly minimal.

This claim is incorrect. Let A consist of ω and finite subsets of ω (denoted $p(\omega)$) along with a relation I, on pairs (i, σ), with $i \in \omega$ and $\sigma \in p(\omega)$, such that $I(i, \sigma) \Leftrightarrow \sigma(i) = 1$, and relations W and Q which pick out ω and $2^{<\omega}$. Then I claim $W(A)$ is minimal. Suppose not. Then there is some formula $\varphi(x, \bar{a})$ such that $W(x) \wedge \varphi(x, \bar{a})$ is neither finite nor cofinite. But it is clear that A is homogeneous, and so, fixing the finitely many elements algebraic over $\bar{a}$, there is an automorphism taking some i not in $\operatorname{acl}(\bar{a})$ with $\varphi(i, \bar{a})$ to some j not in $\operatorname{acl}(\bar{a})$ with $\neg\varphi(j, \bar{a})$, which is impossible. Moreover, $W(A)$ has infinite dimension, since $\operatorname{acl}(\bar{i}) = \bar{i}$, for any tuple $\bar{i} \in W(A)$. $W(x)$ remains minimal with infinite dimension if we take an elementary extension of A by expanding ω to ω_1, and taking all finite subsets of ω_1, since the same arguments go through. Thus, even if the problem means to specify a proper elementary extension, A still has the required properties.

However, $W(x)$ is not strongly minimal. Let B be the elementary extension of A formed by taking the universe to be $\omega \cup 2^{\omega}$. Then $W(B)$ is certainly not minimal, since for any $\sigma \in 2^{\omega}$ such that $\{i \mid \sigma(i) = 0\}$ and $\{i \mid \sigma(i) = 1\}$ are both infinite, $I(x, \sigma)$ defines a non-finite, non-cofinite set in $W(x)$.

9.2.7*. Show that if A is an L-structure and P^A is a minimal set of uncountable cardinality, then A is strongly minimal.

This is false on its face from the example of the previous problem, as modified at the end of the first paragraph. It is also false if we simply take A to be ω_1, with R a unary relation such that $|R(A)| = \omega_1$, $|\neg R(A)| = \omega$, but even if the statement is modified to "P^A is strongly minimal" instead of "A is strongly minimal," the example from the previous problem disproves it.

9.2.8. Show that the structure $(\omega, <)$ is minimal but not strongly minimal.

An easy back-and-forth argument shows that if $\varphi(x, \bar{a})$ has n quantifiers, then if $a = \max(\bar{a})$, for all $b > 2^n + a$, $\varphi(b, \bar{a})$ must have a constant truth value. Thus, $\varphi(\omega, \bar{a})$ is either finite or cofinite. However, let B be a proper elementary extension of $(\omega, <)$, and fix $b \in B \setminus \omega$. It is easy to see that $b > i$ for any $i \in \omega$, and also that $B \models \exists_{\geq n} x(x > b)$ for every $n \in \omega$, both by $(\omega, <) \preccurlyeq B$. But then $x < b$ witnesses that B is not minimal.

9.2.9. Let A be a structure (Ω, E), where E is an equivalence relation whose equivalence classes are all finite, and E has just one class of cardinality n for each positive integer n. Show that A is minimal but not strongly minimal.

The formulas $\exists_{=n} x(xEy)$ $(n < \omega)$, xEy, and $x = y$ form an elimination set for $\operatorname{Th}(A)$. It is easy to see then that any formula defines a finite or cofinite set on A. However, let B be an elementary extension of A. B necessarily has a class of size $\geq \omega$. Then, taking b in that class, xEb defines an infinite co-infinite set.

9.3.1*. Let $\phi(\bar{x})$ be a formula of L with parameters in M. (a) Show that if $\bar{b}$ is any sequence of elements of M, and N is $(M, \bar{b})$, then ϕ has the same Morley rank calculated in N as it has in M. (b) Give an example to show that if $L' \subseteq L$ and ϕ is a formula of L', then the Morley rank of ϕ calculated in $M|L'$ need not be the same as in M.

For (a), go by induction on Morley rank. Suppose $RM_N(\phi) \geq \alpha + 1$. Then we have $\psi_1, \ldots \in L(N)$, pairwise disjoint, with $RM(\phi \wedge \psi_i) \geq \alpha$. But $L(N) = L(M)$, so for each $\phi \wedge \psi_i$, the induction hypothesis applies, and so we are done.

For (b), consider $\phi(x) = x = x$, let L' be the empty language, let $L = \{R\}$, a language with one unary predicate, and let M be a model such that $R(M)$ and $\neg R(M)$ are both infinite. Then ϕ has Morley rank 1 in L' and Morley rank 2 in L.

9.3.2. Write out a full proof of Lemma 9.3.2(a,c). [(a): If $M \models \forall \bar{x}(\phi \rightarrow \psi)$ then $RM(\phi) \leq RM(\psi)$. More generally if $\psi'(\bar{x}')$ is a formula of L with parameters in M and there is a formula of L with parameters in M which defines an injective map from $\phi(M^n)$ to $\psi(M^m)$ for some $m < \omega$, then $RM(\phi) \leq RM(\psi')$. (c): If $(\mathfrak{M}, \bar{a}) \equiv (\mathfrak{M}, \bar{b})$ then $RM(\chi(\bar{x}, \bar{a})) = RM(\chi(\bar{x}, \bar{b}))$.]

(a): By induction on α, show that $RM(\phi) \geq \alpha$ implies $RM(\psi) \geq \alpha$ for any such ϕ and ψ whenever this injective map exists. Fix ϕ and ψ and let $\sigma(\bar{x}, \bar{x}')$ define an injective map (if $\phi \rightarrow \psi$, $\sigma(\bar{x}, \bar{x}') = \bar{x} = \bar{x}'$). At $\alpha = 0$, trivial, and at limits, by induction. We are left with $RM(\phi) \geq \beta + 1$. Then we have $\theta_1, \ldots$, formulas of L with parameters from M, pairwise disjoint, with $RM(\phi \wedge \theta_i) \geq \beta$. Consider $\exists \bar{x}((\phi \wedge \theta_i)(\bar{x}) \wedge \sigma(\bar{x}, \bar{x}'))$. This defines a set in $\psi(M^m)$. It is mapped into injectively by σ from $(\phi \wedge \theta)(\bar{x})$, so by induction, it has Morley rank at least β. Since each i gives a different set, disjoint from the others by σ's injectivity, we have infinitely many pairwise disjoint sets with Morley rank β, and so we are done.

(c): I show $RM(\chi(\bar{x}, \bar{a})) \geq RM(\chi(\bar{x}, \bar{b}))$ – this is enough by symmetry. Go by induction on α, for $RM(\chi(\bar{x}, \bar{a})) \geq \alpha$. At 0, $\mathfrak{M} \equiv \mathfrak{N}$ does it, and at limits, there is nothing to do. So let $RM(\chi(\bar{x}, \bar{a})) \geq \beta + 1$. Then we have $\theta_1(\bar{x}, \bar{b}_1), \theta_2(\bar{x}, \bar{b}_2), \ldots$, formulas with parameters in M, witnessing this. For each θ_i, we have that $RM(\chi(\bar{x}, \bar{a}) \wedge \theta_i(\bar{x}, \bar{b}_i)) \geq \beta$. By ω-saturation of $\mathfrak{M}$, we can find $\bar{b}_i'$ such that $(\mathfrak{M}, \bar{a}) \equiv (\mathfrak{M}, \bar{b}, \bar{b}_i')$. By induction, $RM(\chi(\bar{x}, \bar{b}) \wedge \theta_i(\bar{x}, \bar{b}_i')) \geq \beta$. Extending the elementary equivalence to include all of the $\bar{b}_i$'s, $\mathfrak{M}$ will prove that all of these sets are disjoint, and so $RM(\chi(\bar{x}, \bar{b})) \geq \beta + 1$.

9.3.3*. Let A' be an L-structure and $\phi(\bar{x})$ a formula of L with parameters $\bar{a}$ in A'. Show that there is a set T of sentences of L with parameters $\bar{a}$, such that the following are equivalent, for $(A, \bar{a}) \equiv (A', \bar{a})$. (a) There is a set of formulas $\psi_{\bar{s}}(\bar{x})$ with parameters from some elementary extension B of A, such that the properties (i)-(iv) of Lemma 9.3.1(b) hold with B for A. (b) A is a model of T. [Lemma 9.3.1(b) gives a tree of formulas exhibiting that $RM(\phi(\bar{x})) = \infty$. I changed A to A' in the statement of the problem, since otherwise there is a trivial solution.]

Suppose some model elementarily equivalent to $(A', \bar{a})$ has such an elementary extension, as in (a).

We can assume this model is A', WLOG. For $\bar{s} \in {}^k 2$, let $\bar{s}|-1$ be $\bar{s}|k-1$. If $\bar{s} = (\bar{s}|-1) \frown 0$, let $\bar{s}* = (\bar{s}|-1) \frown 1$, and likewise if $\bar{s} = (\bar{s}|-1) \frown 1$. Let T consist of all sentences of the form

$$\exists \bar{b}_{(0)}, \bar{b}_{(1)}, \bar{b}_{(00)}, \bar{b}_{(01)}, \ldots, \bar{b}_{\bar{s}\frown 0}, \bar{b}_{\bar{s}\frown 1}$$
$$\left(\forall x \left(\bigwedge_{\bar{s} \in {}^{<n}2} \left(\psi_{\bar{s}}\left(\bar{x}, \bar{b}_{\bar{s}}\right) \to \psi_{\bar{s}|-1}\left(\bar{x}, \bar{b}_{\bar{s}|-1}\right)\right) \wedge \left(\psi_{\bar{s}}\left(\bar{x}, \bar{b}_{\bar{s}}\right) \to \neg\psi_{\bar{s}*}\left(\bar{x}, \bar{b}_{\bar{s}*}\right)\right)\right) \wedge \bigwedge_{\bar{s} \in {}^{<n}2} \exists \bar{x} \psi_{\bar{s}}\left(\bar{x}, \bar{b}_{\bar{s}}\right)\right),$$

for some $n \in \omega$, with an existential quantifier for each $b_{\bar{s}}$, $s \in {}^{<n}2$, and $\psi_{\langle\rangle}(\bar{x}, \bar{b}_{\langle\rangle}) = \phi(\bar{x}, \bar{a})$. Then T asserts the existence of finite approximations to the tree described in the Lemma. Since B actually has such a tree, it satisfies T, and thus so does A'. Now, take any A with $(A, \bar{a}) \equiv (A', \bar{a})$. $A \models T$, so by compactness, it has an elementary extension in which the full tree is realized.

Thus, if some model elementarily equivalent to $(A', \bar{a})$ satisfies (a), then every such model does, so (a) implies (b), and (b) implies (a) by the above compactness argument.

9.3.4. Let K and L be first-order languages, A a K-structure and B an L-structure. Suppose B is interpretable in A. Show that if A has Morley rank α, then the Morley rank of B is at most $(\alpha+1)^n$ for some $n < \omega$.

The universe of B is the image of some definable map on n-tuples for some n. All formulas in K defining sets in B can be translated into formulas in L defining sets in A and the property of disjointness is preserved. It is thus trivial to show that $RM(B) \leq RM(A^n) \leq (\alpha+1)^n$.

9.3.5. Let A be an L-structure, X a set of elements of A and $\bar{a}, \bar{b}$ tuples of elements of A. Show that if $p(\bar{x}) = \mathrm{tp}_A(\bar{a}/X)$ has Morley rank α and $\bar{b}$ is algebraic over X and $\bar{a}$, then $q(\bar{y}) = \mathrm{tp}_A(\bar{b}/X)$ has Morley rank $\leq \alpha$.

Let $\varphi(\bar{x}) \in p$ have minimal Morley rank in p. Let $\bar{b}$ be algebraic over $\bar{a}$ with formula θ (with parameters in X), so we have $\chi(\bar{a}, \bar{b}) = \exists_{=n} \bar{y} \theta(\bar{y}, \bar{a}) \wedge \theta(\bar{b}, \bar{a})$. Write $\zeta(\bar{a}, \bar{b}) = \chi(\bar{a}, \bar{b}) \wedge \varphi(\bar{a})$. For any $\bar{a}'$, we know that $RM(\zeta(\bar{a}', \bar{y})) \leq 0$, since there can only be n solutions, if there are any. Then, by Erimbetov's inequality, $RM(\exists \bar{x} \zeta(\bar{x}, \bar{y})) \leq RM(\exists \bar{y} \zeta(\bar{x}, \bar{y}))$. But the second formula is in p and has minimal Morley rank, and the first is in q. Thus $RM(q) \leq RM(p)$.

9.3.6. Give examples to show that for every positive integer d there are a structure A and a complete type $p(x)$ over $\emptyset$ with respect to A, such that p has Morley rank 1 and Morley degree d.

Let A be a model with one equivalence relation, and d classes, all infinite. Then there is only one 1-type over $\emptyset$, $x = x$, but it can be broken into the disjoint equivalence classes, each of which is defined by a formula with a parameter from that class.

9.3.7. Let $p(x)$ be a complete type over a set X with respect to the L-structure A. Suppose $\phi(x)$ and $\psi(x)$ are formulas of L with parameters in X, and suppose $\phi \in p$. Let $\theta(x, y)$ be a formula with parameters in X, which defines in A a bijection from $\phi(A)$ to $\psi(A)$. Show that there is a complete type $q(y)$ over X, such that for every formula $\sigma(y)$ of L with parameters in X, $\sigma \in q$ if and only if $\exists y(\theta(x,y) \wedge \sigma(y))$ is in p. Show that q has the same Morley rank as p.

Define $q = \{\sigma(y) \mid \exists y(\theta(x,y) \wedge \sigma(y)) \in p\}$. We show q is complete and consistent. First, consistency. Suppose not. Then we have $\sigma_1(y), \ldots, \sigma_n(y) \vdash \neg\sigma_0(y)$, for some $\sigma_i(y) \in q$ $(i \leq n)$, so we have $\neg\sigma_0(y) \vee \ldots \vee \neg\sigma_n(y)$. Let a realize p in some extension of A, B. Then we have $\exists y(\theta(a,y) \wedge \sigma_i(y))$ $(i \leq n)$. θ remains a bijection, since it is a definable property, but since θ is a bijection, there is only one b with $\theta(a,b)$, so we must have $\sigma_i(b)$ $(i \leq n)$, which is impossible. Thus, q is consistent. Now for completeness. Since p is complete, it must either contain $\exists y(\theta(x,y) \wedge \sigma(y))$ or $\neg\exists y(\theta(x,y) \wedge \sigma(y))$, for each $\sigma(y)$. Since for any realization of $p(x)$, there certainly is a (unique) element satisfying $\theta(x,y)$, σ must be false on it. But then $\neg\sigma(y)$ is true on it, so we have $\exists y(\theta(x,y) \wedge \neg\sigma(y))$. Thus q is complete, and so it is a type.

Any formula in q (after it is conjuncted with ψ) defines a set that is mapped injectively (through θ) to a set defined by a formula in p, and vice versa. Thus, through Lemma 9.3.2(a), $RM(q)$ must equal $RM(p)$.

9.3.8. Let G be a group which is totally transcendental. Prove: (a) there are no infinite strictly decreasing chains of definable subgroups of G. [Consider cosets and use Lemma 9.3.1.] (b) Every intersection of a family of definable subgroups of G is equal to the intersection of a finite subfamily.

For (a), suppose there is an infinite strictly decreasing chain of definable subgroups, $H_0 \geq H_1, \ldots$. Each coset aH_i is definable, and so we easily get a tree as in Lemma 9.3.1, since every group has at least 2 cosets in the previous group. For (b), if not, then the descending chain condition would be violated by the intersections.

9.4.1. Show that in the definition of 'unstable theory' it makes no difference if we allow parameters in the formula ϕ of (4.1).

Suppose we have such a ϕ with parameters, so write ϕ as $\phi(\bar{x}, \bar{y}, \bar{c})$. We have $\{\bar{a}_i \mid i < \omega\}$ fulfilling (4.1) for ϕ. Write $\bar{c} = (c_1, \ldots, c_k)$ and $\bar{x} = (x_1, \ldots, x_n)$. Let $\bar{u} = (u_1, \ldots, u_{k+n})$, and likewise for $\bar{v}$. Now take $\psi(\bar{u}, \bar{v}) = \phi(u_1, \ldots, u_n, v_1, \ldots, v_n, u_{n+1}, \ldots, u_{n+k})$. Then $\{\bar{a}_i \frown \bar{c} \mid i < \omega\}$ witnesses (4.1).

9.4.2. Show that if $\mathbb{Q}$ and $\mathbb{R}$ are respectively the field of rational numbers and the field of reals, then both $\mathrm{Th}(\mathbb{Q})$ and $\mathrm{Th}(\mathbb{R})$ are unstable.

The formula $\varphi(x,y)$ defined by "$y-x$ is a non-zero sum of four squares" orders the positive integers in both $\mathbb{R}$ and $\mathbb{Q}$.

9.4.3. Show that every complete theory with the strict order property is unstable.

We have a formula ϕ which orders arbitrarily long finite chains in each model of T. By compactness, there is a model of T where ϕ orders an infinite chain.

9.4.4. Show that every complete theory with the independence property is unstable.

Let $\phi(\bar{x},\bar{y})$ be a formula with the independence property. By compactness, find a model with $\{\bar{b}_i \mid i<\omega\}$ witnessing ϕ's independence. Then for each k, we can find $\bar{a}_k$ with $\phi(\bar{a}_k,\bar{b}_j) \Leftrightarrow j \leq k$. Defining $\psi(\bar{y},\bar{x}) = \phi(\bar{x},\bar{y})$, $\{(\bar{b}_i,\bar{a}_i) \mid i<\omega\}$ witnesses that ψ is an unstable formula, and thus that T is unstable.

9.4.5. Give examples of (a) a complete theory which has the strict order property but not the independence property, (b) a complete theory which has the independence property but not the strict order property.

The theory of dense linear orders without endpoints certainly has the strict order property. It does not have the independence property by a result due to Shelah/Vapnik-Chervonenkis. The theory of the random graph easily has the independence property. Suppose it had the strict order property. Then there would be a definable partial ordering on n-tuples of the random graph, for some n, $\varphi(\bar{x},\bar{y})$. But the theory of the random graph has quantifier elimination. So we can write any formula ψ as a disjunction of conditions of the form $\theta(\bar{x},\bar{y}) = \bigwedge_{i\in u, j\in v} x_iRy_j \wedge \bigwedge_{i\in w,j\in z} \neg x_iRy_j$. Now choose $\bar{a}_1,\bar{a}_2$ with $\phi(\bar{a}_1,\bar{a}_2)$. We can then inductively find $\bar{b}=(b_1,\ldots,b_n)$ such that $\theta(\bar{a}_2,\bar{b})$ for some θ like the above in ϕ, and $\theta(\bar{b},\bar{a}_1)$ for some θ like the above in $\neg\phi$. But then $\bar{b}$ contradicts ϕ being a partial ordering.

9.4.6. Show that if T is unstable it has an unstable formula.

If T is unstable, it has some formula of the form $\varphi(\bar{x},\bar{y})$, with $\{\bar{c}_i \mid i<\omega\}$ witnessing (4.1). Let $\psi(\bar{x},\bar{y})$ be $\bar{x}=\bar{y} \vee \phi(\bar{x},\bar{y})$. Then ψ is an unstable formula, witnessed by $\{(\bar{c}_i,\bar{c}_i) \mid i<\omega\}$.

9.4.7. Let L be a first-order language and T a complete theory in L. (a) Show that if $\phi(\bar{x},\bar{y})$ is a stable formula for T, then so is $\neg\phi(\bar{x},\bar{y})$. (b) Show that if $\phi(\bar{x},\bar{y})$ and $\psi(\bar{x},\bar{y})$ are stable formulas for T, then so is $(\phi\wedge\psi)(\bar{x},\bar{y})$.

For (a), we prove the contrapositive. By compactness we have an infinite sequence $\{(\bar{a}_i,\bar{b}_i) \mid i<\omega\}$, with $\models \neg\phi(\bar{a}_i,\bar{b}_j) \Leftrightarrow i\leq j$. But then for any n we can find $(\bar{c}_1,\bar{d}_1),\ldots,(\bar{c}_n,\bar{d}_n)$ such that $\models \phi(\bar{c}_i,\bar{d}_j) \Leftrightarrow i\leq j$ (set $(\bar{c}_i,\bar{d}_i) = (\bar{a}_{n-i},\bar{b}_{n-i-1})$). Then ϕ is unstable.

For (b), again show the contrapositive. By compactness, we have $\{(\bar{a}_i,\bar{b}_i) \mid i<\omega\}$, with $\models (\phi\wedge\psi)(\bar{a}_i,\bar{b}_j) \Leftrightarrow i\leq j$. Define a 3-coloring of $[\omega]^2$ by $f(i,j)=0$ if $(\neg\phi\wedge\neg\psi)(\bar{a}_j,\bar{b}_i)$, $f(i,j)=1$ if $\neg\phi(\bar{a}_j,\bar{b}_i)$, and $f(i,j)=2$ if $\neg\psi(\bar{a}_j,\bar{b}_i)$. Note that both ϕ and ψ cannot be true, because that would contradict $\phi\wedge\psi$ being unstable. By Ramsey's theorem, we can find an infinite subset of ω, K, such that f is constant on $[K]^2$. But then $\{(\bar{a}_i,\bar{b}_i) \mid i\in K\}$ will witness either ϕ's or ψ's instability.

9.4.8. Suppose $\phi(\bar{x},\bar{y})$ is a stable formula for T. (a) Show that if $\bar{y}'$ is a tuple of variables which include all those in $\bar{y}$, and $\psi(\bar{x},\bar{y}')$ is equivalent to $\phi(\bar{x},\bar{y})$ modulo T, then ψ is also stable. (b) Show that if $\bar{x}'$ is $\bar{y}$ and $\bar{y}'$ is $\bar{x}$, and $\theta(\bar{x}',\bar{y}')$ is the formula $\phi(\bar{x},\bar{y})$, then θ is stable.

Doing the contrapositive for (a), with compactness, we have $\{(\bar{a}_i, \bar{b}_i') \mid i < \omega\}$ witnessing ψ's instability. Then the appropriate $\bar{b}_i$ replacing $\bar{b}$ witnesses ϕ's instability.

For (b), the contrapositive and compactness gives us $\{(\bar{a}_i, \bar{b}_i) \mid i < \omega\}$ witnessing θ's instability. Then a similar argument to 9.4.7(a) gives that ϕ is unstable.

9.4.9. Show that every stable integral domain is a field.

We show any integral domain not a field is unstable. Let t be any non-zero non-invertible element. Consider the sequence $1, t, t^2, \ldots$. The relation $\exists z(xz = y)$ orders this sequence if it is infinite, and makes the integral domain unstable. If the sequence is finite, then for some $m, n \in \omega$, $t^m = t^n$. By cancellation, t is invertible.

9.4.10. Show that if T is an unstable complete theory in the first-order language L, then there is a formula $\phi(x, \bar{y})$ of L such that for arbitrarily large finite n, T implies that there are $\bar{a}_0, \ldots, \bar{a}_{n-1}$ for which $\neg\exists x \bigwedge_{i<n} \phi(x, \bar{a}_i)$ holds, but $\exists x \bigwedge_{i \in w} \phi(x, \bar{a}_i)$ holds for each proper subset w of n.

We first show that $(\mathbb{Q}, <)$ has the above property. Define $\phi(x, y_0, y_1, y_2, y_3) = y_0 < x < y_1 \vee y_1 < x < y_2$. Now, for any n, let $\bar{a}_i = (i, n+i-1, n+i, 2n+i-1)$ $(0 \leq i < n)$. We verify that this choice works. First check that not all the $\phi(\mathbb{Q}, \bar{a}_i)$'s can be simultaneously satisfied. Suppose some rational k does. If $k < i$ $(i < n)$, then $k \notin \phi(\mathbb{Q}, \bar{a}_i)$. Thus, $k \geq n-1$. But $[n-1+i, n+i]$ is not in $\phi(\mathbb{Q}, \bar{a}_i)$, which implies $k > 2n-1$, but $[2n-1, \infty)$ is not in $\phi(\mathbb{Q}, \bar{a}_0)$, so k cannot exist. Now check that omitting any $\phi(\mathbb{Q}, \bar{a}_i)$ makes the remainder satisfiable: any point in $(n-1+i, n+i)$ will be in every other set. Thus, $(\mathbb{Q}, <)$ has Shelah's finite cover property.

By compactness, if T is unstable, we can find a model in which some formula φ defines a dense linear ordering without endpoints on some countable set. Then applying the above argument yields Shelah's finite cover property.

9.4.11. Let the L-structure A be a model of a stable theory T, let $p(\bar{x})$ be a complete type over X with respect to A, and let $\phi(\bar{x}, \bar{y})$ be a formula of L. Define the *strict ϕ-rank* of $p(\bar{x})$ to be the minimum value of $\mathrm{BI}(\phi, \psi)$ as $\psi(\bar{x})$ ranges over all formulas of p which are conjunctions of formulas of the form $\phi(\bar{x}, \bar{c})$ or $\neg\phi(\bar{x}, \bar{c})$. Prove Theorem 9.4.9 [definability of types] using strict ϕ-rank in place of ϕ-rank.

Following the proof, the strict ϕ-rank of p is some finite number n (perhaps greater than the ϕ-rank), witnessed by some ψ of the above form. Let $\mathrm{d}_p\, \phi(\bar{y})$ be the formula '$\mathrm{BI}(\phi, \psi \wedge \phi(-, \bar{y})) \geq n$'. Now, if $\phi(\bar{x}, \bar{c}) \in p$, then $\mathrm{BI}(\phi, \psi \wedge \phi(-, \bar{c})) \geq n$, since $\psi \wedge \phi(\bar{x}, \bar{c})$ is in the required form. Inversely, if not, then $\neg\phi(\bar{x}, \bar{c}) \in p$, and since $\psi \wedge \neg\phi(\bar{x}, \bar{c})$ is in the required form, $\mathrm{BI}(\phi, \psi \wedge \neg\phi(-, \bar{c})) \geq n$, so by Lemma 9.4.11, $\mathrm{BI}(\phi, \psi \wedge \phi(-, \bar{c})) < n$, so $\neg\, \mathrm{d}_p\, \phi(\bar{c})$.