## 1 Introduction

My research is in o-minimality, a branch of model theory with connections to real analysis and algebraic geometry, since o-minimal structures can be seen as generalizations of the real numbers. My results have applications in analysis and ordered algebraic structures. I completed my dissertation at the University of California, Berkeley in December 2008 under T. Scanlon, have three published/accepted papers in model theory, three in preparation/submitted, and an additional two published in biomathematics.

Analysis is a natural area in which to apply o-minimality [Spe99], but o-minimality has also been used in a proof of some new cases of the André-Oort conjecture [Pil11], and in the theory of Lie groups, in the proof of a conjecture of Barbasch and Vogan [SV00]. Formally, an o-minimal structure is a totally ordered structure in which every one-dimensional first-order definable set is a finite union of points and intervals [vdD98]. The reals as a field form an o-minimal structure [vdD98]. It remains o-minimal when the exponential function is added [Wil96], and even after adding all analytic functions with compact domain [vdDMM94]. An o-minimal field (that is, an o-minimal structure expanding a field) is necessarily real closed.

A recent result of mine (§2) gives a complete description of linear orders definable in o-minimal fields – essentially, they are lexicographically-ordered cartesian powers of the field. Another recent result from an ongoing collaboration with C. Steinhorn (§2.1) says that partial orders can be constructively extended to linear orders in a wide variety of contexts (not only o-minimality). Both these results have connections to research in theoretical economics on preference relations – orders on cartesian powers of the reals.

Another recent result (§4), joint with K. Peterzil and P. Eleftheriou, is part of a program in o-minimality to connect real Lie groups to groups in o-minimal structures. We show that any interpretable group in an o-minimal structure is actually definable and a real Lie group after quotienting by a model-theoretic analogue of the identity component. This yields a unified framework for treating interpretable groups. Interpretable groups in o-minimal structures include $PGL_n(R), PSU_n(R)$, etc.

My main dissertation result (§3) concerned a problem roughly described as: given a curve and a continuous function on a neighborhood of the curve, find a compact neighborhood of the curve onto which the function can be extended continuously. The curve might be defined using analytic functions, while the function in question and the desired neighborhood are algebraic. In the o-minimal context, I give necessary and sufficient conditions on a curve such that, given any function, such a neighborhood exists. This question arose from the study of differential equations in o-minimal structures. I have two other dissertation results (both published) [Ram10c, Ram10b].

I have several current research projects. One is in the realm of analysis (§5). With M. Edmundo, J.-P. Rolin, and T. Servi, I am part of a research project on quasi-analytic algebras of real-valued functions, with connections to the "Dulac problem with parameters." Another is with C. Steinhorn (§2.2). We are working on describing precisely for which structures our result on partial orders hold, and on a full classification of partial orders definable in o-minimal structures, as well as applications of our results to economics. Finally, I am working with K. Peterzil and P. Eleftheriou to continue and extend our work on interpretable groups, and interpretable sets in general (§4.1). We hope to finish the proof of Pillay's Conjecture, a longstanding program in model theory, and also to prove elimination of imaginaries for arbitrary interpretable sets in o-minimal structures.

## 2 Classifying Orders

My latest accepted paper [Ram11], to appear in the Proceedings of the American Mathematical Society, concerns linear orders definable in o-minimal structures. An o-minimal structure comes equipped with $<$, a linear ordering, and my result shows that $<$ completely controls all other orders. It improves a prior result of Onshuus and Steinhorn [OS09, Cor. 5.1].

**Theorem 2.1.** *Let $M$ be an o-minimal field and let $(P, \prec)$ be a definable linear order with $n = \dim(P)$. Then there exists a definable embedding $g$ of $(P, \prec)$ into $(M^{n+1}, <_{lex})$, where $<_{lex}$ is the lexicographic order on $M^{n+1}$. Moreover, $g(P) \subseteq M^{n+1}$ has finite projection to the last coordinate.*

"Definable" here means given by a first-order formula in the language of $M$. Thus, if $M$ is just $\mathbb{R}$ with field structure, then $e^x$ is not a definable function, but $x^{3/2}$ is. (Although by [Wil96], in this case $M$ would remain o-minimal if we added $e^x$ as a definable function.)

Theorem 2.1 says that any definable linear order in an o-minimal field is definably isomorphic to a subset of a lexicographic order. Thus, definable linear orders are very easy to classify and to analyze, since the lexicographic order is well-understood, and any definable order can be taken to be a subset of a lexicographic order.

Onshuus and Steinhorn's result [OS09, Cor. 5.1] says, in the context of Theorem 2.1, that $P$ can be partitioned into finitely many sets, each of which embeds in a lexicographic order of the same dimension. However, the whole order is not fully classified, since the ordering of elements in different sets in the partition is not determined. Theorem 2.1 strengthens this result by avoiding the partition. Thus, to compare any two elements, we can just map them to the lexicographic order and then compare them lexicographically.

### 2.1 From Linear to Partial Orders

Theorem 2.1 completely classifies linear orders in appropriate o-minimal structures, but partial orders are less-understood. Hasson and Onshuus [HO10] show that any partial order definable in an o-minimal structure actually interprets a linear order. This result, while remarkable, is only local – it gives no information about the global structure of the partial order. The following question is due to J. Truss [MS97]:

**Question 2.2.** *Let $(P, \prec)$ be a partial order definable in an o-minimal structure, $M$. Does there exist $\prec'$, an $M$-definable total order on $P$ that extends $\prec$.*

A $\prec'$ extending $\prec$ always exists, assuming the axiom of choice. What is highly non-trivial is that it should be definable in the same o-minimal structure. Macpherson and Steinhorn [MS97] gave a positive answer to this question when $P$ is one-dimensional, but the general case remained open. However, recently C. Steinhorn and I have proven the following theorem:

**Theorem 2.3.** *Let $(P, \prec)$ be a partial order definable in a weakly o-minimal structure. Then there is a definable $\prec'$, a total order on $P$ extending $\prec$.*

Since all o-minimal structures are weakly o-minimal, this gives a positive answer to Question 2.2. Our proof transforms the problem in a novel way, so that we need only consider partial orders coming from the subset relation on definable families of sets. This approach allows us to use induction on the o-minimal dimension. Not only does our method yield a positive answer to Question 2.2, it gives a constructive method for defining the linear order.

### 2.2 Active Work

The techniques in Theorem 2.3 are not limited to the weakly o-minimal case, but apply to many "tame" ordered settings. We have two conditions on $M$, either of which implies that every partial order can be definably extended to a total order. With A. Usvyatsov and C. Steinhorn, I am investigating where these conditions fit with other model-theoretic concepts, like dp-minimality, local o-minimality, and o-minimalism.

C. Steinhorn and I are working on an even finer classification of partial orders in o-minimal theories, with the aid of Theorems 2.1 and 2.3. As with linear orders, many partial orders studied throughout mathematics and the sciences are definable in o-minimal structures, and hence a greater understanding of these partial orders may aid in a wide variety of areas.

### 2.3 Applications to Economics

C. Steinhorn is working on the connections between economics and o-minimality, and he and I plan to work together to extend and investigate further applications of our results in o-minimality to economics. He noted that economists study linear orders extensively, under the name of "preference relations." A preference relation is just an order on states of a system, with more favorable states ordered above less favorable ones. Once the relation has been quotiented out by "indifference curves" (states which are equivalent in the ordering), we arrive at a linear order. Most preference relations considered in economics are on subsets of $\mathbb{R}^n$ [Moo10, p. 11], with the relations real analytic, piecewise-linear, or, in general, definable in an o-minimal structure. C. Steinhorn pointed out that thus questions on preference relations can be generalized to o-minimal structures, and resolved there.

Spash and Hanley [SH95] discuss the issue of "mixed" lexicographic preferences in economics, when preferences are locally lexicographic, but the order changes. For instance, let $X$ represent an individual's income, and $Y$ the well-being of an endangered species. When $X$ is above a certain threshold, an individual may always prefer a state where $Y$ is larger, even at the cost of $X$.

However, for $X$ small, they may prefer greater $X$ at the cost of a much smaller $Y$. Theorem 2.1 shows that this scenario can actually be modeled simply as a lexicographic order, with all "threshold" testing encapsulated in a single transformation function.

Moreover, economists also study "incomplete preference relations," in which two states may be neither equivalent nor comparable. In other words, partial orders. C. Steinhorn and I plan to apply our result on completing partial orders to resolve some widely considered problems in economics on incomplete preference relations (see, e.g. [Dug99]).

## 3 Extending Functions on Curves

The bulk of my dissertation [Ram08] (submitted as [Ram10a]) was devoted to proving the following theorem. It answered a question of P. Speissegger, about extending a bounded function to a closed set containing a (not necessarily definable) curve. It arose for him in the study of differential equations, specifically trying to prove an analogue to a theorem of Malgrange [Mal74] in which the existence of a formal solution to an ordinary differential equation in a space of generalized series implies that there is actually a $C^\infty$ function that is a solution, and that has asymptotic expansion this formal solution.

**Theorem 3.1.** *Let $M$ be an o-minimal field. Let $\gamma$ be a non-oscillatory curve in $M^n$ with one endpoint $0$. Then the following two statements are equivalent:*

1. *For every definable function, $f$, bounded on some initial segment of $\gamma$, there is a definable set, $C$, containing an initial segment of $\gamma$ such that $f$ is continuous on $C$ and extends continuously to $\mathrm{cl}(C)$.*

2. *Let $\bar{c} = \langle c_1, \ldots, c_n \rangle$ be any "infinitesimal" point of $\gamma$, reordered to be decreasing (the required reordering is independent of choice of $\bar{c}$). The type of $c_i$ over $M(\bar{c}_{<i})$ is not near scale or in scale on $M$, for $i = 1, \ldots, n$.*

"Scale" and "decreasing" are technical but easy concepts. Roughly, the scale of an element is how close the image of a definable function approaches it. A tuple $\langle c_1, \ldots, c_n \rangle$ is decreasing if for each $i < n$, the element $c_i$ is "maximal" in some sense over all $c_j$ with $j > i$. Given a curve, it is straightforward to reorder it to be decreasing, and then to examine the scale of each coordinate.

## 4 Interpretable Groups

A group is definable in an o-minimal structure if it can be regarded as a definable set with the group operation a definable function. Definable groups have been a major object of study in o-minimality for many years. A prominent conjecture of Pillay was that such groups are isomorphic to Lie groups after quotienting by a model-theoretic equivalent of the identity component of the group [Pil04], with the group and Lie group having the same dimension, suitably calculated. Pillay's Conjecture has been verified for o-minimal structures expanding ordered fields [HPP08], and ordered groups [Pet09], but remains open when the structure does not expand a group, and when the group being considered is interpretable.

Interpretable groups are those in which the underlying set is a definable set quotiented by a definable equivalence relation, and the function acts on equivalence classes. For example, projective linear groups are interpretable, with the equivalence relation equality up to scalar multiplication.

Joint work of mine with K. Peterzil and P. Eleftheriou (to be submitted as [EPR11]) proves:

**Theorem 4.1.** *Let $M$ be o-minimal. Every group interpretable in $M$ is definably isomorphic to a definable group in $M$. Moreover, this definable group is a subset of a cartesian product of one-dimensional definable groups.*

Thus, Pillay's Conjecture for interpretable groups is no harder than Pillay's Conjecture for definable groups. In fact, our result proves the first part of Pillay's Conjecture – an interpretable group quotiented out by its identity component is a real Lie group.

Our proof uses techniques from a broad spectrum of o-minimality. We develop a group topology on interpretable groups, and do analyses of definably simple groups, definably compact groups, abelian groups, etc. We also develop the theory of general interpretable sets in o-minimality. Finally, we prove results like those in [PS98], constructing a group from a binary increasing function.

Our work is most relevent in o-minimal structures not expanding groups, which do occur in this area – in fact, in the proof of Pillay's Conjecture for groups definable in o-minimal fields, it is shown that the Lie group is definable in an o-minimal structure that does not *a priori* expand an ordered group [HP09]. In such structures, interpretable groups were not known to be definable before our work, and Pillay's Conjecture remains open.

### 4.1 Active Work

K. Peterzil, P. Eleftheriou, and I are applying our result to the remaining case of Pillay's Conjecture, when the o-minimal structure does not expand a group. Since we can embed a definable group in a cartesian product of one-dimensional groups, we can use the tools developed in the proof of Pillay's Conjecture in the group case. We believe that all we are lacking is a theory of cohomology and torsion points as in [EO04]. Going through their arguments and adapting them to the case we are in, where there is no global group but a finite number of groups instead, will give us Pillay's Conjecture in full generality, and complete an 8-year program in model theory.

On another front, we are investigating to what extent o-minimal structures have full elimination of imaginaries. This means that given any equivalence relation, we can uniformly pick a distinct representative for each equivalence class, thus allowing us to treat the set of classes as an ordinary set, eliminating the relation. Many o-minimal structures eliminate imaginaries, but it is not known that elimination can be done in general. Our conjecture is that any equivalence relation can be eliminated, after naming finitely many parameters. This would, of course, generalize our result about interpretable groups, and more broadly mean that any object interpretable in an o-minimal structure could be taken to be definable instead, simplifying its study. Our strategy uses the equivalence relation to find local groups around each point, and then uses those groups to make definable choices.

## 5 Quasi-Analytic Functions

Some of o-minimality's greatest successes have come by proving that certain analytic structures are o-minimal, thus opening up these structures to the power of o-minimal techniques. I am part of a joint project, with J.-P. Rolin of the University of Bourgogne, and M. Edmundo and T. Servi, both with me at CMAF, University of Lisbon, to analyze algebras of quasi-analytic functions and show their o-minimality. Quasi-analytic functions share with analytic functions the property that to each function is associated a unique formal power series. However, in the quasi-analytic case, the power series is an asymptotic expansion that may not converge to the function. Such functions arise in many contexts in real analytic geometry. For instance, when examining analytic vector fields in $\mathbb{R}^2$, the Poincaré return maps that occur will be quasi-analytic.

We plan to show the o-minimality of $\mathbb{R}$ with appropriate algebras of quasi-analytic functions, and then, using o-minimality, to prove strong results on these algebras, such as a preparation theorem (à la Weierstrass), and through that, quantifier elimination. By quantifier elimination, we would have fine control over the functions in our algebra, and in fact the goal is for this quantifier elimination to be constructive. Hence, given an analytic vector field in $\mathbb{R}^2$, we would be able to explicitly construct the Poincaré return maps as quasi-analytic functions, and fully understand them.

My particular role in this project is in the proof of a preparation theorem for these structures. There have already been powerful preparation theorems proved for various o-minimal structures [vdDS02], but they are highly non-constructive. Through careful analysis of these proofs, I plan to extract a constructive technique to prepare functions. For a function $F(x,t)$, with $t$ a single variable and $x$ a tuple, a preparation consists of a finite partition of its domain such that on each set in the partition, we have

$$F(x,t) = (t-\theta(x))^\lambda a(x)u(x,y), \qquad |u(x,y)-1| \leq \frac{1}{2},$$

for some functions $\theta, a, u$ and constant $\lambda$. The proofs rely on model-theoretic types, and I plan, using classifications like those in the proof of Theorem 3.1, to simplify this so that it can be understood exactly what properties are needed. For instance, if $\lim_{t\to b} F(x,t) = 0$ for some fixed $x$ and $b$, then in a neighborhood of $b$, we already understand how to prepare $F$, by theorems for o-minimal polynomially-bounded structures. The preparation theorem of [vdDS02] uses this implicitly, and so I can constructivize the proof in this case.

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